2002

2002-1. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a polynomial of degree D. Find the maximal number of connected components and the maximal number of closed components of the *parabolic curve* $\mathrm{Par}(f)$ of its graph (where $f_{xx}f_{yy} = f_{xy}^2$):

$$b_0(\mathrm{Par}(f)) = ?, \quad b_1(\mathrm{Par}(f)) = ?.$$

Even for $D = 4$, it is not known whether b_1 attains the value 4*, and the constants C in the lower and the upper bounds for large degrees D, $b_1 \sim CD^2$, differ by a factor of order of* 4:

$$(D-1)(D-2)/2 \le b_1 \le (2D-5)(D-3)+1.$$

2002-2. Let $M \subset \mathbb{R}\mathrm{P}^3$ be a *smooth algebraic surface* of degree D. Find the maximal number of connected components of its parabolic line.

The constants C in the lower and the upper bounds CD^3 differ by a factor of order of 20:

$$D(D-1)(D-2)/2 \le b_0 \le 10D^3 - 28D^2 + 4D + 3.$$

The lower estimates in problems 2002-1 and 2002-2 mean the existence of surfaces with many closed parabolic curves.

2002-3. Let $f:\mathbb{S}^1 \to \mathbb{R}$ be a smooth function; it is called *D-hyperbolic* if the second differential $d^2 f$ of the homogeneous function $f(x,y) = r^D F(\varphi)$ (where $x = r\cos\varphi$, $y = r\sin\varphi$) is hyperbolic (has signature $(+,-)$) everywhere in $\mathbb{R}^2 \setminus \{0\}$.

Find the connected components of the space of D-hyperbolic functions: is the index (equal to the number of rotations of the cross $d^2 f = 0$ when the point (x,y) makes one revolution around the origin) the unique invariant of the connected component? *The set of the values attained by the index is infinite and unbounded below* (*but bounded above*).

2002-4. For the polynomial case (where F is a trigonometric polynomial and f is an ordinary homogeneous polynomial of degree D), find the number of connected components of the set of D-hyperbolic polynomials. Is it growing linearly with D when the latter is high?

2002-5. Consider a controlled dynamical system $\dot{x} = v(x,u)$, where x is a point of a compact phase manifold M and u belongs to a compact controlling parameter manifold U. Let $f : M \to \mathbb{R}$ be a smooth goal function.

Study the *mean optimization problem*, maximizing the time average $\widehat{f} = \lim_{T\to+\infty} T^{-1} \int_0^T f(x(t))\,dt$ by a clever choice of the control $u(t)$ (and eventually of the initial state $x(0)$).

If the problem (i. e., the pair formed by v and f) depends generically on some exterior parameters, then the optimization strategy and the optimal average might have singularities ("phase transitions") at the points of a *hypersurface of phase transitions* in the manifold P of the values of the exterior parameter.

Find the *generic phase transitions in the mean optimization problem*, at least when the dimensions of the manifolds M, U, P are not large. *The problem is open even when all these manifolds are* 1*-dimensional where there are already some nontrivial stable singularities* (*see the paper* ARNOLD V. I. Optimization in mean and phase transitions in controlled dynamical systems. *Funct. Anal. Appl.*, 2002, **36**(2), 83–92).

2002-6. Let $f : M \to \mathbb{R}$ be a smooth function on a compact Riemannian manifold, and $0 < r < R < \infty$ be two smooth functions on M. Study the *mean value optimization problem* for the space average

$$\widehat{f} = \left(\int_M f(x)\rho(x)\,dx\right) \Big/ \left(\int_M \rho(x)\,dx\right)$$

for the mass distribution defined by a density function ρ with respect to the Euclidean volume element dx provided that this density is restricted by the inequalities $r \le \rho \le R$ everywhere on M.

Study the generic phase transitions for the case where f, r, and R depend smoothly on exterior parameters.

It is known that the optimal strategy is $\{$*to choose* $\rho = r$ *where* $f(x) < c$ *and* $\rho = R$ *where* $f(x) > c$ *for some constant* $c\}$*, but the study of phase transitions requires the investigation of the influence of some strange logarithmic singularities and of their regularizations in the case of even-dimensional manifolds M, as in many physical problems. See the paper* ARNOLD V. I. On a variational problem related to the phase transitions of the averages in controlled dynamical systems. In: Nonlinear Problems in Mathematical Physics I. In honour of Professor O. A. Ladyzhenskaya. Editors: M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov and N. N. Ural′tseva. Dordrecht: Kluwer Acad. Publ., 2002, 23–34 (Internat. Math. Ser., 1).

2002-7. Let $u_0 : M^2 \to \mathbb{R}$ be a smooth "initial" function on a Riemannian manifold M (the case of a 2-dimensional ball B^2 is already relevant). Study the *minimization problem for the Dirichlet integral* $\int_M (\nabla u)^2\, dx$, where the function u is obtained from the initial function u_0 by an area-preserving diffeomorphism of M onto itself (by an "*incompressible fluid motion*").

The extremal function u is smooth if the smooth initial mountain $u_0 : B^2 \to \mathbb{R}$, *vanishing on the boundary of the ball, has just one nodegenerate* (*Morse*) *maximum inside the ball. In this case, the extremal function u is the symmetrization of* u_0 (*depending only on the distance from the center of the ball*).

But for the initial smooth mountain u_0 *having* (*like the Elbrus mountain*) *two local maxima separated by a saddle point, the extremal function seems to have a singularity of type* $|x|$ *along a curve with unknown extremal function singularities at its endpoints. The problem is to study such singularities for generic* u_0.

2002-8. The (C,B,A)-*permutation* of the set $\{1,2,\dots,n\}$ transports to the last place the subset $A = \{1,2,\dots,a\}$ preceded by the transported set $B = \{a+1, \dots, a+b\}$ while the starting position is occupied by $C = \{a+b+1, \dots, n\}$.

Some of these $(n-1)(n-2)/2$ permutations permute *cyclically* (like the addition of a constant to the residues $\bmod\, n$), and some of these cyclic permutations are *transitive* (like the addition of the constant 1).

Find the proportion of both the cyclic and the transitive cyclic permutations among the (C,B,A)-permutations for large n.

More generally, starting from a permutation of k elements, one defines a permutation of the set $\{1,\dots,n\}$ from its decomposition into k segments $\{a_i + 1, \dots, a_{i+1} - 1\}$. The problem is to study the statistics of the Young diagrams formed by the cycle lengths of the resulting permutations, for the case of large n and random decompositions of n into k parts.

2002-9. A mapping $\mathbb{C}^n \to \mathbb{C}^n$ (or $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$) is called a *pseudocomplex* mapping if it sends complex subspaces to complex subspaces (one may consider separately the cases of vector, affine or projective subspaces—all the three versions are interesting).

A real diffeomorphism $\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^2$ is pseudocomplex if and only if either it, or its product with the complex conjugation, is a complex projective mapping (and similarly for the other versions and other n's).

Do there exist other pseudocomplex homeomorphisms? Other pseudocomplex bijections?

These questions should have been studied by Hilbert as a part of axiomatic projective geometry, but his school seems to have missed these foundational problems.

2002-10. To formulate quaternionic versions of the questions in problem *2002-9*, one should distinguish the left subspaces and the right subspaces. I would suggest studying those mappings which send left and right subspaces onto left and right subspaces (with a left one sent onto a right one also permitted).

2002-11. The complexification and quaternionization paradigm had been used by me many times starting from its invention in ARNOLD V. I. Distribution of ovals of real plane algebraic curves, involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms. *Funct. Anal. Appl.*, 1971, **5**(3), 169–176; *the Russian original is reprinted in*: Vladimir Igorevich Arnold. Selecta–60. Moscow: PHASIS, 1997, 175–187 (see, for instance, ARNOLD V. I. Polymathematics: is mathematics a single science or a set of arts? In: Mathematics: Frontiers and Perspectives. Editors: V. I. Arnold, M. Atiyah, P. Lax and B. Mazur. Providence, RI: Amer. Math. Soc., 2000, 403–416, and ARNOLD V. I. Symplectization, complexification and mathematical trinities. In: The Arnoldfest. Proceedings of a conference in honour of V. I. Arnold for his sixtieth birthday (Toronto, 1997). Editors: E. Bierstone, B. A. Khesin, A. G. Khovanskiĭ and J. E. Marsden. Providence, RI: Amer. Math. Soc., 1999, 23–37 (Fields Inst. Commun., 24)).

For instance, it is now proved that the complex version of the tetrahedron is the octahedron: ${}^{\mathbb{C}}A_3 = B_3$, see ARNOLD V. I. Complexification of tetrahedron and pseudoprojective transformations. *Funct. Anal. Appl.*, 2001, **35**(4), 241–246.

Now the problem is to prove my old conjecture that its quaternionic version is the icosahedron:

$$ {}^{\mathbb{H}}A_3 = H_3, \quad {}^{\mathbb{C}}B_3 = H_3. $$

Perhaps one should start with the easier plane versions:

$$ {}^{\mathbb{C}}A_2 = B_2, \quad {}^{\mathbb{H}}A_2 = H_2, \quad {}^{\mathbb{C}}B_2 = H_2 $$

relating the symmetry groups of the triangle, the square, and the pentagon.

The difficulty of all this subject lies in its nonmathematical character: *the problem is to find the definition of the informal quaternionization operation rather than to prove any ready mathematical statement.*

2002-12. The *caustic of a periodic function* $g : \mathbb{S}^1 \to \mathbb{R}$ is the curve in the plane $\mathbb{R}^2$ of the functions

$$ G_{A,B} : \mathbb{S}^1 \to \mathbb{R}, \quad G_{A,B}(\varphi) = g(\varphi) + A\cos\varphi + B\sin\varphi, $$

consisting of those functions which are not Morse:

$$ \{(A,B) \in \mathbb{R}^2 : \ \exists\varphi : G'_{A,B}(\varphi) = G''_{A,B}(\varphi) = 0\}. $$

The caustics of generic periodic functions have many peculiar properties: for instance, each caustic has at least four cusps, and its alternative length (the alternating sum of the lengths of its segments between the cusps) vanishes. The cusps of a caustic having just 4 cusps form a parallelogram (and the barycenters of the odd and of the even cusps coincide if there are more than 4 cusps).

The problem is to replace smooth periodic functions in this theory with exact Lagrangian submanifolds of the phase cylinder $T^*\mathbb{S}^1$. Such a submanifold corresponding to a function is the graph of its differential. A general Lagrangian submanifold needs not be a section of the cotangent bundle, and the graph of the corresponding "*multivalued potential*" function needs not be an immersed curve: it may have cusps.

It is interesting to understand, which would be the four-cusp property version for the caustics of such exact Lagrangian submanifolds, and what would happen to the Sturm–Hurwitz theorem on the zeros of Fourier series (being the infinitesimal version of the caustics' cusps theorem) for such extended "multivalued periodic functions."

2002-13. The theory of caustics of periodic functions and Lagrangian submanifolds discussed in problem *2002-12* depends on the functions $x = \cos\varphi$ and $y = \sin\varphi$ on the circle $x^2 + y^2 = 1$. Replacing the circle with a different curve, say, with an algebraic curve C: $f(x,y) = 0$, and g with the restriction to C of a function on the plane, say, of a polynomial $P(x,y)$, we define the *C-caustic* (as the curve of non-Morse restrictions of the functions $P + Ax + By$ to C).

The problem is now to extend the Sturm–Hurwitz theorems on Fourier series (as well as their extensions described in problem *2002-12* and, in more detail, in the works ARNOLD V. I. Astroidal geometry of hypocycloids and the Hessian topology of hyperbolic polynomials. *Russian Math. Surveys*, 2001, **56**(6), 1019–1083; Moscow: Moscow Center for Continuous Mathematical Education Press, 2001 (in Russian)) to the C-caustics associated with more general curves C than the circle used in problem *2002-12*.

2002-14. Study the triangulations of the torus $\mathbb{T}^2$ associated with cubic algebraic number fields (by the theory of higher-dimensional continued fractions). One starts with a matrix $A \in \mathrm{SL}(3,\mathbb{Z})$ having 3 positive eigenvalues. The 3 corresponding invariant planes divide $\mathbb{R}^3$ into 8 invariant octants. Each open octant contains the semigroup of its integer points. The boundary of the convex hull of this set of integer points, $\mathbb{Z}^3 \cap$ octant, is called the *sail*. The sail is invariant under A. It is an (infinite) polyherdal surface whose faces are bounded by convex compact polygons. It had been proved (by Dirichlet and H. Tsuchihashi—see TSUCHIHASHI H. Higher-dimensional analogues of periodic continued fractions and cusp singularities. *Tôhoku Math. J., Ser.* 2, 1983, **35**(4), 607–639) that the sail is invariant under the action of the commutative subgroup $\mathbb{Z}^2$ of $\mathrm{SL}(3,\mathbb{Z})$ formed by the matrices with the same eigenvalues.

The torus dealt with in this problem is the quotient space

$$\mathbb{T}^2 = (\text{the sail of } A)/\mathbb{Z}^2.$$

It is divided into the images of the faces of the sail under factorization. Each image contains some "integer points" (the images of the integer points of the face). Thus, we have associated a geometric object to A: a decomposition of $\mathbb{T}^2$ into "convex polygons" containing "integer points."

The problem is to understand which decompositions of $\mathbb{T}^2$ (and which sets of "distinguished integer points") can be obtained in this way from various matrices A.

2002-15. While comparing problem *2002-14* with the situation for *ordinary continued fractions* (where there are no restrictions on "a triangulation of the torus" and on "its distinguished points," since every finite sequence of integers is a period of some quadratic irrational number), I must repeat the old interesting problem (see problem *1993-11*) to compare the statistics of elements of the period of a random irrational number with that of the eigenvalues of a random matrix in $\mathrm{SL}(2,\mathbb{Z})$ with real eigenvalues.

I mean first to consider integer points (p,q) defining quadratic equations $x^2+px+q=0$ whose roots are real, such that $p^2+q^2 \le N$. Among the elements of the period of the continued fractions of the roots of this equation, we consider the proportion of 1's, of 2's, and so on. The limit of the proportion of k's for N tending to infinity is called the *k's statistic* for random quadratic irrational numbers.

The statistic for periods of the eigenvalues of random matrices is defined similarly, starting with those integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for which $ad-bc=1$, the eigenvalues are real and $a^2+b^2+c^2+d^2 \le N$.

2002-16. The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on the Lobachevskian plane as well as on the de Sitter world (the exterior domain of the disk of the Klein model for the Lobachevskian plane), see problem *1996-15* and book ARNOLD V. I. Arithmetics of Binary Quadratic Forms, Symmetry of Their Continued Fractions, and Geometry of Their de Sitter World. Moscow: Moscow Center for Continuous Mathematical Education Press, 2002; *Bol. Soc. Brasil. Mat.* (*N. S.*), 2003, **34**(1), 1–42. The orbits of this group are discrete in the Lobachevskian plane (but accumulate to the "absolute" circle) and are everywhere dense in the de Sitter world.

a) How precisely does an orbit inside the "absolute" circle accumulate to this circle?

b) How is an orbit outside the "absolute" circle distributed? Is there any kind of ergodicity and equipartition, as for the products of 2 rotations of the sphere or of the plane (see problems *1963-6–1963-12* for the details)?

The relation of the Lobachevsky and de Sitter geometry to the arithmetic and albegra of groups $\mathrm{SL}(2,\mathbb{Z}_p)$ is described in the paper ARNOLD V. I. Fermat dynamics, matrix arithmetics, finite circles and finite Lobachevsky planes. *Funct. Anal. Appl.*, 2004, **38**(1), 20 pp., where, for instance, the finite Lobachevky plane $\bmod p$ and "upper" half-plane containing $p(p-1)/2$ points are treated.

2002-17. The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on the set $\mathbb{Z}^3$ of binary quadratic forms mx^2+ny^2+kxy with integer coefficients m, n, k. The number $h(D)$ of orbits of this action on the set of the forms with a fixed negative value of the determinant $D=4mn-k^2$ is finite, see Theorem 13 in the book ARNOLD V. I. Arithmetics of Binary Quadratic Forms, Symmetry of Their Continued Fractions, and Geometry of Their de Sitter World. Moscow: Moscow Center for Continuous Mathematical Education Press, 2002. Explore the function $h(D)$. What is the asymptotic behavior of $h(D)$ as $D\to-\infty$?

2002-18. A binary quadratic form $f(x,y)=mx^2+ny^2+kxy$ with integer coefficients m, n, k is said to be *perfect* if the set $S=f(\mathbb{Z}^2)\subset\mathbb{Z}$ of the values of this form on $\mathbb{Z}^2$ is a multiplicative semigroup (i. e., $uv\in S$ whenever $u\in S$ and $v\in S$). Perfect quadratic forms were studied in the book ARNOLD V. I. Arithmetics of Binary Quadratic Forms, Symmetry of Their Continued Fractions, and Geometry of Their de Sitter World. Moscow: Moscow Center for Continuous Mathematical Education Press, 2002. What is the probability that a randomly chosen binary quadratic form with integer coefficients is perfect? If this form is perfect, what can one say about the structure of the semigroup of its values? *The product of three values is always a value, as it was proved in the book quoted above.*

2002-19. If positive integers a and $n>1$ are mutually prime then $a^{\varphi(n)}\equiv 1 \mod n$ where $\varphi(n)$ is the number of positive integers that are less than n and mutually prime with n (the Euler theorem). Explore the following problem: For what divisors N of $\varphi(n)$ does the relation $a^{\varphi(n)/N}\equiv 1 \mod n$ take place? The relation $a^{\varphi(n)/N}\equiv -1 \mod n$? The case $a=2$ (and n odd) is already highly nontrivial.

2002-20. Let two positive integers a and n be mutually prime. To what extent is the sequence $t\mapsto a^t \mod n$ ($t\geq 1$ integer) random?

2002-21. Examine the sequence $\varphi(n)/n$ ($n > 1$ integer) where the Euler function $\varphi(n)$ is the number of positive integers that are less than n and mutually prime with n.

According to Gauss, the probability of that two randomly chosen integers are mutually prime is equal to $6/\pi^2$. This implies that $\varphi(n)/n$ tends to $6/\pi^2$ as $n \to \infty$ in a certain weak sense [which is also confirmed by calculations of $\varphi(n)/n$ for not so large n]. Determine a rigorous meaning of this statement and prove it. What are the "oscillations" and the "variance" (and other probabilistic characteristics) of the sequence $\varphi(n)/n$?

2002-22 (the Fermat–Euler dynamical system)**.** Let n be a large odd integer, and let $\Gamma = \Gamma(n)$ be the set of the $\varphi(n)$ residues $\operatorname{mod} n$ that are mutually prime with n, φ being the Euler function.

The doubling mapping $x \mapsto 2x$ acts on Γ with N orbits of equal lengths, $l = \varphi(n)/N$. Is the set of l residues forming one orbit asymptotically random if n becomes large?

For a truly random finite sequence of l elements of a set of m elements, the absence of any repetition seems rather probable for short sequences, where l^2 is small with respect to m, and seems rather improbable for long sequences, l^2 being large with respect to m. Indeed, the number of choices of l distinct elements is $C = m(m-1)(m-2)\cdots(m-l+1)$, while the total number of the unrestricted choices is $T = m^l$. Hence, $C/T = \prod_{k=0}^{l-1}(1-k/m)$, $\ln(C/T) = \sum_{k=0}^{l-1}\ln(1-k/m) \sim -\sum_{k=0}^{l-1} k/m \sim -l^2/2m$.

Thus, the ratio of the orbit period $l = \varphi(n)/N$ and the number $\varphi(n)$ of all the possible values of elements of the orbit might indicate the randomness degree of the geometric progression $\{2^i\} \operatorname{mod} n$: large values of l^2 with respect to $\varphi(n)$ (i. e., the smallness of the divisor $N(n)$ of $\varphi(n)$ with respect to the square root of n) would indicate some nonrandomness.

The calculations of $N(n)$ for odd integers $n \leq 512$ show rather an average linear growth (say, $N(511) = 48$ while $\varphi(511) = 432$).

For more details on the Fermat–Euler dynamical system see: ARNOLD V. I. Fermat–Euler dynamical systems and the statistics of arithmetics of geometric progressions. *Funct. Anal. Appl.*, 2003, **37**(1), 1–15; ARNOLD V. I. Ergodic and arithmetic properties of geometric progression's dynamics. *Moscow Math. J.*, 2004, to appear; ARNOLD V. I. Euler Groups and the Arithmetic of Geometric Progressions. Moscow: Moscow Center for Continuous Mathematical Education Press, 2003 (in Russian); ARNOLD V. I. Topology and statistics of formulae of arithmetics. *Russian Math. Surveys*, 2003, **58**(4), 637–664; ARNOLD V. I. The topology of algebra: combinatorics of squaring. *Funct. Anal. Appl.*, 2003, **37**(3), 177–190; ARNOLD V. I. Fermat dynamics, matrix arithmetics, finite circles and finite Lobachevsky planes. *Funct. Anal. Appl.*, 2004, **38**(1), 20 pp.