2001

2001-1 (A. Ortiz-Rodriguez)**.** Given a real polynomial f in two variables x and y, denote by $P(f)$ the set of parabolic points on the surface $\{z = f(x,y)\}$, i. e., the zero set of the Hessian $H[f] = f_{xx}f_{yy} - f_{xy}^2$. Determine the maximal number of

a) compact connected components,

b) all the connected components

of the set $P(f)$ over all the polynomials f of given degree d. How can these connected components be mutually arranged? The first case where the answer is unknown is $d = 4$.

The Hessian $H[f]$ *of a polynomial* f *of degree* d *is a polynomial of degree* $\le m = 2d-4$. *The Harnack inequality ensures that the parabolic set* $P(f)$ *has at most* N *compact connected components, where*

$$N = \frac{(m-1)(m-2)}{2} + 1 = (d-3)(2d-5) + 1.$$

For general polynomials of degree $2d-4$, *this estimate is attained. However, it is not clear whether this estimate is attained for polynomials of degree* $2d-4$ *that are Hessians. The problem is the simplest case of* Hessian topology.

There are examples of polynomials f *of degree* d *for which the number of compact connected components of* $P(f)$ *is at least*

$$\frac{(d-1)(d-2)}{2}.$$

So, for d *large, the maximal number of compact connected components of* $P(f)$ *lies asymptotically between* $d^2/2$ *and* $2d^2$. What is the true asymptotic of this number?

Similar questions on the parabolic curves are also open for such surfaces in $\mathbb{R}^3$ as the graphs of rational functions and for the graphs of the odd degree roots of real polynomials in two variables, as well as for the graphs of other single-valued real algebraic functions of a fixed degree d.

2001-2 (A. Ortiz-Rodriguez)**.** Given a smooth algebraic surface $M \subset \mathbb{R}\mathrm{P}^3$, denote by $P(M)$ the set of parabolic points on M. Determine the maximal number of

a) connected components of the set $P(M)$ diffeomorphic to $\mathbb{S}^1$,

b) all the connected components of the set $P(M)$

over all the smooth surfaces M of given degree d. How can these connected components be mutually arranged?

This problem is a generalization of the previous one. It is known that the number of connected components of $P(M)$ *diffeomorphic to* $\mathbb{S}^1$ *is at most*

$$10d^3 - 28d^2 + 4d - 3.$$

On the other hand, there are examples of surfaces M *of degree* d *for which the number of connected components of* $P(M)$ *diffeomorphic to* $\mathbb{S}^1$ *is at least*

$$\frac{d(d-1)(d-2)}{2}.$$

So, for d *large, the maximal number of connected components of* $P(M)$ *diffeomorphic to* $\mathbb{S}^1$ *lies asymptotically between* $d^3/2$ *and* $10d^3$. What is the true asymptotic behavior of this number?

2001-3. Let D be a real number and (r,φ) polar coordinates in the real plane. Denote by $\mathrm{Hyp}(D)$ the set of smooth functions $F:\mathbb{S}^1 \to \mathbb{R}$ such that the homogeneous function $f(r,\varphi) = r^D F(\varphi)$ of degree D is hyperbolic, i. e., its second quadratic form $d^2 f$ is of signature $(+,-)$ everywhere for $r > 0$.

For $D \geq 0$ *integer,* f *is a homogeneous polynomial of degree* D *in* $x = r\cos\varphi$, $y = r\sin\varphi$ *if and only if* F *is a trigonometric polynomial of degree* D *and* $F(\varphi+\pi) \equiv (-1)^D F(\varphi)$.

Determine the connected components a) of the set $\mathrm{Hyp}(D)$ b) of the subset $\mathrm{Hyp}_{\mathrm{Pol}}(D)$ of $\mathrm{Hyp}(D)$ corresponding to f polynomial (for $D \geq 0$ integer).

The set $\mathrm{Hyp}_{\mathrm{Pol}}(4)$ *is connected* (V. I. Arnold, F. Aicardi)*, while the set* $\mathrm{Hyp}_{\mathrm{Pol}}(6)$ *consists of at least two connected components* (ARNOLD V.I. Astroidal geometry of hypocycloids and the Hessian topology of hyperbolic polynomials. *Russian Math. Surveys*, 2001, **56**(6), 1019–1083; Moscow: Moscow Center for Continuous Mathematical Education Press, 2001 (in Russian)). *Conjecturally, the number of connected components of* $\mathrm{Hyp}_{\mathrm{Pol}}(D)$ *grows like* $\mathrm{const}\cdot D$ *as* $D \to \infty$. *The set* $\mathrm{Hyp}(D)$ *of smooth functions has infinitely many connected components. In the polynomial case, even the number of connected components of the subset* $\mathrm{Hyp}_{\mathrm{Pol}}(D)$ *is unknown, already for* $D = 6$.

2001-4. Let $g:\mathbb{S}^1 \to \mathbb{R}$ be a smooth function. Its *caustic* is by definition the plane curve

$$C = \left\{(A,B) \in \mathbb{R}^2 \;\middle|\; \text{the function } \varphi \mapsto g(\varphi) + A\cos\varphi + B\sin\varphi \text{ is non-Morse}\right\}$$

(see ARNOLD V. I. Astroidal geometry of hypocycloids and the Hessian topology of hyperbolic polynomials. *Russian Math. Surveys*, 2001, **56**(6), 1019–1083; Moscow: Moscow Center for Continuous Mathematical Education Press, 2001 (in Russian)). Recall that $G:\mathbb{S}^1 \to \mathbb{R}$ is said to be *non-Morse* if there exists a point

$\phi \in \mathbb{S}^1$ such that $G'(\phi) = G''(\phi) = 0$. *For instance, for* $g(\varphi) = \cos(2\varphi)$ *the caustic* C *is the astroid*

$$A = -4\cos^3\phi, \quad B = 4\sin^3\phi \qquad (\phi \in \mathbb{S}^1).$$

In this parametric equation of C*,* ϕ *is just the point where both the first and second derivatives of* $\cos(2\varphi) + A\cos\varphi + B\sin\varphi$ *vanish.*

What curves on $\mathbb{R}^2$ are the caustics of periodic functions? *That* C *is a caustic imposes some restrictions on the curve* C:

1. *A caustic has at least* 4 *cusps.*
2. *The number of cusps is even.*
3. *The alternated length of a caustic* (*we change sign after each cusp*) *is zero.*
4. *Through any point of the plane, there pass at least two tangents to the caustic.*
5. *A caustic possesses no inflection points.*

One can also show that the caustic of a trigonometric polynomial is an algebraic curve of genus zero (*see the paper cited above*)*.*

The problem is to describe the set of restrictions complete in the following sense: each curve satisfying those restrictions is a caustic.

This problem can be generalized in several directions. First, one may consider the so-called *hypercaustic* in $\mathbb{R}^{2n}$, i. e., the curve

$$\mathcal{C} = \Bigg\{ (A_1, \dots, A_n, B_1, \dots, B_n) \in \mathbb{R}^{2n} \,\Bigg|\, \text{the function} \\ G\colon \varphi \mapsto g(\varphi) + \sum_{k=1}^{n} \big[A_k\cos(k\varphi) + B_k\sin(k\varphi)\big] \text{ has a critical point } \phi \\ \text{where the derivatives } G' = G'' = \cdots = G^{(2n)} = 0 \text{ all vanish} \Bigg\}.$$

Second, instead of the circle $\mathbb{S}^1$ and trigonometric polynomials

$$\sum_k \big[A_k\cos(k\varphi) + B_k\sin(k\varphi)\big],$$

one can consider respectively an arbitrary curve $\Gamma \subset \mathbb{R}^2$ and polynomials on $\mathbb{R}^2$ restricted to Γ.

Apart from that, it is also possible to consider exact Lagrangian submanifolds in $T^*\mathbb{S}^1$ in place of functions (a closed curve $L \subset T^*\mathbb{S}^1$ is called an exact Lagrangian submanifold if the difference between L and the zero section is the boundary of a chain of area zero).

2001-5. Set

$$\Sigma^{2n-1} = \left\{ (A_1,\ldots,A_n,B_1,\ldots,B_n) \in \mathbb{R}^{2n} \,\middle|\, \text{the trigonometric polynomial} \right.$$
$$\left. \cos[(n+1)\varphi] + \sum_{k=1}^{n} \big[A_k\cos(k\varphi) + B_k\sin(k\varphi)\big] \ \text{ is non-Morse} \right\}. \quad (1)$$

The discriminant Σ divides $\mathbb{R}^{2n} = \{(A_1,\ldots,A_n,B_1,\ldots,B_n)\}$ into $n+1$ domains G_2, G_4, $\ldots$, G_{2n+2} according to the number of critical points of polynomial (1). Explore the topology and the singularities combinatorics of these domains.

The domain G_{2n+2} of trigonometric M-polynomials (1) (*in the terminology of I. G. Petrovskiĭ*) *was examined in the paper* ARNOLD V. I. Topological classification of real trigonometric polynomials and cyclic serpents polyhedron. In: The Arnold–Gelfand Mathematical Seminars: Geometry and Singularity Theory. Editors: V. I. Arnold, I. M. Gelfand, V. S. Retakh and M. Smirnov. Boston, MA: Birkhäuser, 1997, 101–106; *the Russian translation in*: Vladimir Igorevich Arnold. Selecta–60. Moscow: PHASIS, 1997, 619–625. *This particular domain has a convex polyhedral model* (*simply a square for* $n = 1$). *It is conjectured that all these domains have polyhedral models in terms of the affine Coxeter group mirrors, similar to the descriptions of the swallowtails pyramids polyhedral models in terms of the Springer cones decompositions into the Weyl chambers for the linear Coxeter group case. But this conjecture is not confirmed yet even for small values of n.*

2001-6. Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function, $h(R) > 0$ for $R > 0$ and $h(0) = 0$. Consider a curve F on $\mathbb{R}^2$ with a semicubic cusp O. Denote by ℓ_P the part of the normal to F at point P where F is smooth containing the center of curvature. Let R_P be the radius of curvature of F at P. Let Π_P be the parabola with vertex P and axis ℓ_P whose radius of curvature at the vertex is equal to $h(R_P)$.

Study the envelope of the family of the parabolas $\{\Pi_P\}$. *If F is an astroid and $h(R) = \frac{2}{3}R$, then the family $\{\Pi_P\}$ has a smooth envelope which is tangent to F at cusp O.* Does the family of the parabolas $\{\Pi_P\}$ possess a smooth envelope for other curves F (for, possibly, other functions h)? *If the envelope is smooth at O, it is tangent to F there.*

Similar problems are also interesting for the families of generic smooth curves instead of the parabolas (of curves having the same properties of the tangency to F and of the curvature radius at the tangency points).