2000

2000-1 (A. Ortiz-Rodriguez)**.** How many parabolic curves (closed curves or all the curves—these are two different questions) can lie on the graph of a real polynomial of degree D in two variables? This is unknown even for $D = 4$ (is it possible that there are 4 closed components?).

2000-2 (A. Ortiz-Rodriguez)**.** How many parabolic curves can lie on a projective algebraic surface of degree D in $\mathbb{R}\mathrm{P}^3$: even the asymptotics for large D is of interest (the coefficients at D^3 in the examples I know and in the known upper estimate differ by a factor of 20).

2000-3. Consider the space of hyperbolic [with the second differential of signature $(+,-)$ everywhere except at the origin] homogeneous polynomials of degree D in two real variables. How many connected components does this space consist of? (*For* $D = 3$ *or* 4 *there is only one component, for* $D = 6$ *there are at least two components*; the conjectural answer grows probably with a linear rate as D increases, i. e., the number of the components is of the order of D for D large.)

2000-4. Consider a generic collection of n straight lines in $\mathbb{R}\mathrm{P}^2$. How much does the number of topological classes of such collections differ from the number of topological types of collections of n noncontractible circles embedded generically in $\mathbb{R}\mathrm{P}^2$?

Similar questions are not trivial even for the affine plane, both in the case of embeddings of affine straight lines and in the case of circles—in the presence of a fixed number of intersections as well as even without intersections.

Of course, the question makes sense for straight lines in the three-dimensional space too, provided that the complexity of topological knotting of the curve configurations to be compared is bounded above.

2000-5. The observers assert that the number of the eruptions of the volcano of *Piton de la Fournaise* with the emission of volume less than V grows like $V^{-3/2}$ as V decreases [Lahaie F., Grasso J.-R., Marcenac P., Giroux S. Modélisation de la dynamique auto-organisée des éruptions volcaniques: application au comportement du Piton de la Fournaise, Réunion. *C. R. Acad. Sci. Paris, Sér.* IIa *Sci. Terre Planètes*, 1996, **323**(7), 569–574]. Are there reasonable grounds for this scaling law, similarly to the turbulence laws?

2000-6. The observers assert that the metabolic rate in similar organisms (such as men of different stature) is proportional to the $3/4$ power of the mass (rather than to the $2/3$ power, as the ratio of the reaction surface area to the reaction volume suggests). Are there reasonable explanations for such a fractal behavior [WEST G. B., BROWN J. H., ENQUIST B. J. A general model for the origin of allometric scaling laws in biology. *Science*, 1997, **276**(5309), 122–126]?

2000-7. There are observations that the number of the species (of animals, insects, birds, ...) on an island of area S is proportional to $S^{1/4}$, whereas the number of the cell types in an organism with the genome of N genes grows with N like $N^{1/2}$. How can one explain these exponents? Compare with the Kolmogorov law, according to which the radius of the minimal but still typical brain or computer of N elements grows like $N^{1/2}$ (rather than like $N^{1/3}$, as the volume argument suggests).

2000-8. Let a mapping of a complex projective space (or vector space) onto itself send all the complex subspaces to complex subspaces. Are there such transformations other than complex projective ones (linear ones) and their products with the complex conjugation?

There are no other diffeomorphisms, but I do not know the answer for the case of homeomorphisms (*hopefully, there are no other homeomorphisms as well*). *One may ask the same question even for the set-theoretic bijections* (*which are not forced to be homeomorphisms*).

2000-9. Let $\Gamma \subset \mathbb{R}^2$ be a real algebraic plane curve and $g : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial. To this pair, assign the *caustic* which is a curve C in another plane equipped with orthonormal coordinates (A,B). The caustic consists of the points (A,B) for which the restriction of the function

$$G_{A,B} = g + Ax + By$$

(x and y being the coordinates in $\mathbb{R}^2$) to the curve Γ possesses a degenerate critical point. For a nonsmooth curve Γ given by the equation $f(x,y) = 0$, the critical points are defined as the zeros of the derivative ∇G, while the degenerate critical points are the zeros of both ∇G and the second derivative $\nabla^2 G$; here ∇ is the Hamiltonian vector field

$$f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y}.$$

If Γ is a circle ($x^2 + y^2 = 1$) then the caustic has at least 4 cusps, and its alternated length (the sum of the lengths of the segments between the cusps with alternating signs) vanishes. This follows from the Sturm–Hurwitz theorem which states that the number of zeros of the sum of a real Fourier series

$$F(t) = \sum_{n>k} \left[a_n \cos(nt) + b_n \sin(nt)\right]$$

is at least the number of zeros of the lowest harmonics entering the series with a nonzero coefficient (i. e., at least $2k+2$ zeros over the period). For instance, if the integral of F vanishes ($k = 0$) then there are at least two zeros (moreover, these zeros are the critical points of the primitive of F). This Sturm theorem proved by Hurwitz is a generalization of the Morse inequality (for the circle), because the function F in the theorem can be viewed as the image of a (primitive in the extended sense) periodic function H under a differential operator of degree $2k+1$:

$$F = LH$$

where

$$L = \partial(\partial^2+1)(\partial^2+4)\cdots(\partial^2+k^2), \quad \partial = (d/dt).$$

Thus, one can regard the zeros of the function F as generalized critical points of the "potential" $H : \mathbb{S}^1 \to \mathbb{R}$.

The problem is to carry over the Sturm–Hurwitz theorem (and the statements on the properties of the caustic) to the case of algebraic curves Γ other than a circle. How many singular points of the caustic are inevitable for curves Γ of a given genus? This question arises even for singular curves Γ of genus zero, e. g., for the degenerate elliptic curve $y^2 = x^2 + x^3$.

2000-10. Consider a controlled dynamical system $\dot{x} = v(x, u)$ on a compact phase space ($x \in M$) with a compact manifold of the values of the controlling parameter u. Let $f : M \to \mathbb{R}$ be a smooth "goal function."

Explore the phase transitions of the controls optimal on the average (i. e., those maximizing the temporal mean

$$\widehat{f} = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(x(t))\, dt$$

—either for the fixed initial point $x(0)$ or while maximizing over this parameter as well).

A *phase transition* here is defined as a nonsmooth dependence (of both the optimal strategy and the attained maximal value of the mean) on additional parameters on which the initial data of the problem (i. e., the controlled system v and the goal function f) depend smoothly.

Such nontrivial phase transitions are encountered even in the simplest one-dimensional case where $M = \mathbb{S}^1$ and $u \in \mathbb{S}^1$.

2000-11. Study the phase transitions of the maximal mean value $\widehat{f}[\rho]$ of a smooth goal function $f : M \to \mathbb{R}$ over the choices of the mass distribution $\rho\,dx$ (with density ρ with respect to the Riemannian volume dx) on M, under the condition that the density is bounded above and below by given positive smooth functions:

$$0 < r(x) \leq \rho(x) \leq R(x) < \infty$$

on M. Here the mean value is defined by the formula

$$\widehat{f}[\rho] = \left(\int_M f\rho\,dx\right) \Big/ \left(\int_M \rho\,dx\right).$$

2000-12. Given an integer matrix A of order three with determinant 1 [$A \in \mathrm{SL}(3,\mathbb{Z})$], construct three eigenplanes assuming that all the eigenvalues are real, positive, and irrational. The integer points in one of the octants bounded by these three planes constitute a commutative semigroup in $\mathbb{R}^3$ while their convex hull is bounded by an infinite polyhedral surface whose vertices are integer (this surface is called the *sail* of the corresponding cubic irrational numbers).

The symmetry group of the sail in $\mathrm{SL}(3,\mathbb{Z})$ has been proved to be $\mathbb{Z}^2$, so that the quotient of the sail by the action of these symmetries turns out to be a two-torus divided into the images of the faces of the sail under the factorization (moreover, on each face that is a convex integer polygon, there were integer points which define distinguished points on the torus as well).

The problem is to calculate explicitly (e. g., using a computer and perhaps the data on cubic irrationalities published by B. N. Delone, D. K. Faddeev, and others) these torus triangulations with the images of the integer points upon them—e. g., for the first hundred of not so large matrices. The simplest example is the matrix $\left(\begin{smallmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ of the "three-dimensional golden section," the conventional golden section corresponds to the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.[1]

The interest of this "experimental" activity is due to the hope of noticing, in the result tables, some regularities which can become theorems in the sequel—for instance, on the statistics of such triangulation properties as the amount of triangular faces and other faces, the proportions of the integer lengths of the edges, those of the numbers of the edges with a common vertex, and so on. Then one would be able to compare such statistics with analogous statistics for other triangulations, e. g., for the sails of random octants or for the convex hulls of the sets of all the integer points in the domains bounded by random smooth surfaces, even by large spheres or ellipsoids. One may also compare the results with the partitions of the plane into the "Voronoĭ polygons" of random (arbitrary or integer) points: a

[1] The greater of the eigenvalues of this 2×2 matrix is $\phi + 2$, where $\phi = \left(\sqrt{5} - 1\right)/2$ is the golden section number.

Voronoĭ polygon of such a system of points is constituted by all the points on the plane for which the nearest point of the system is fixed.

By the way, while averaging in this problem, one can count the contributions of different polygons to the mean either with equal weights (which leads to an unjustifiably large contribution of small polygons since there are plenty of them) or with weights proportional to the polygon areas (which seems more reasonable to me).

Moreover, besides the distributions of the areas, the perimeter lengths, and the numbers of the vertices of the polygons (or the numbers of the sail edges with a common vertex), their joint distributions and correlations are also of interest, as well as the distributions of dimensionless parameters, e. g., the ratio of the area to the perimeter length squared (and the correlation between this ratio and the number of the vertices of the polygon).