1999

1999-1. Compile a *complete* list of the adjacencies of simple curve singularities in $\mathbb{C}^N$.

This and the following six problems are concerned with complex curves, that is, germs $(\mathbb{C},0) \to (\mathbb{C}^N,0)$; however, the same questions for real curves, that is, germs $(\mathbb{R},0) \to (\mathbb{R}^N,0)$, also make sense.

1999-2. Compile a list of the *semigroups* of simple curve singularities in $\mathbb{C}^N$.

a) Does a semigroup determine the type of a (simple) singularity?

b) What pairs of semigroups exclude the adjacency of the corresponding singularities (probably, simplicity is not essential here)?

c) Are the remaining adjacencies realized for some pair of singularities (simple? not simple?) with given semigroups?

1999-3. Is it true that the *simple* curve singularities in $\mathbb{C}^N$ are precisely those *stably simple* singularities that can be realized in $\mathbb{C}^N$?

1999-4. Compile a list of the filtered *Artin algebras* of simple singularities for the curves $f:(\mathbb{C},0) \to (\mathbb{C}^N,0)$.

a) Does such a filtered algebra (or its action on $\mathfrak{m}^1/\mathcal{A}_f$ by operators) determine the type of a simple singularity (or its semigroup)? Here, $\mathfrak{m}^1$ is the maximal ideal in the space of germs of functions $(\mathbb{C},0) \to (\mathbb{C},0)$ and $\mathcal{A}_f$ is the ideal generated by the components of the mapping f.

b) Does the semigroup of a singularity determine its Artin algebra or the filtration?

1999-5. *Resolution of singularities* of simple curves in $\mathbb{C}^N$.

a) Compile a list of resolution graphs. How are they related to question a) in problem *1999-2*?

b) Is it true that moduli of curves arise precisely when moduli of resolutions do (in the case of 4 points on P^1, etc.)?

1999-6. *Stabilization of curves.* Consider the base $\mathbb{C}^{M(N)}$ of versal deformation of a more complex singularity containing the stratum Σ of a simpler singularity.

a) How many (locally) irreducible components does the stratum Σ have?

b) In what sense do the topological (homological? homotopy?) properties of the complement $\mathbb{C}^{M(N)} \setminus \Sigma$ stabilize as $N \to \infty$?

c) In what sense do these properties of the complement (whether or not it stabilizes as $N \to \infty$) stabilize when the type of the simpler singularity is fixed and the type of the initial more complex singularity (simple? any?) becomes more complicated?

1999-7. *The stratum* $\mu = \mathrm{const}$ *for curves*. Consider the "manifold" of singularities of given codimension μ of the orbit in the function space as a submanifold in the base $\mathbb{C}^{\mu}$ of its versal deformation.

a) Is this "manifold" smooth? irreducible?

b) How can its dimension m in $\mathbb{C}^{\mu}$ ("number of internal moduli") be evaluated (with the use of the semigroup? algebra? resolution?) or at least estimated?

c) Is it true that m is semicontinuous with respect to the choice of the initial singularity, i. e., coincides with its usual modality?

1999-8. Fix a positive integer $n \geq 3$ and consider n positive integers $a_1, a_2, \ldots, a_n$. Their sums (linear combinations with integer non-negative coefficients) constitute the semigroup $S(a)$ of positive integers:

$$S(a) := \{\langle k, a\rangle \mid k \in \mathbb{Z}_+^n\}$$

($\mathbb{Z}_+ = \{0; 1; 2; \ldots\}$). Suppose that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Then, starting from certain $K(a) \in \mathbb{Z}_+$, all the non-negative integers lie in $S(a)$. *For instance,* $K(a) = (a_1 - 1)(a_2 - 1)$ *for* $n = 2$. *Note that this value of* $K(a)$ *is always even* (*since the numbers* a_1 *and* a_2 *are relatively prime they cannot be even simultaneously*). *The problem of calculating* $K(a)$ *for* n *large is called the* Frobenius problem.

Explore the statistics of $K(a)$ for typical large vectors a. *Conjecturally,*

$$K(a) \approx c \sqrt[n-1]{a_1 a_2 \cdots a_n}, \quad c = \sqrt[n-1]{(n-1)!}\,.$$

The subsequent three problems are devoted to the statistics of the semigroups of positive integers $S(a)$ for relatively prime $a_1, a_2, \ldots, a_n$ as well. *All these problems are intended mainly for a computer experiment—with the prospects of concluding with proofs. The case* $n = 3$ *is already interesting.*

1999-9. *For* $n = 2$, *a number* $N \in \mathbb{Z}$ *belongs to the semigroup* $S(a)$ *if and only if the number* $K(a) - 1 - N$ *does not* (*J. J. Sylvester*). *Thus, for* $n = 2$ *the semigroup* $S(a)$ *occupies precisely one half of the segment* $[0; K(a) - 1]$ (*recall that, for* $n = 2$, *the number* $K(a) - 1$ *is always odd*).

Determine what fraction of the segment $[0; K(a)-1]$ is occupied by the semigroup $S(a)$ for $n \geq 3$ and for large vectors a. *Conjecturally, this fraction is asymptotically equal to* $1/n$ (*with overwhelming probability for large* a).

1999-10. *Examples show that* $S(a)$ *fills the right half of the segment* $[0; K(a)-1]$ *more densely.*

Find the typical density of filling the segment $[0; K(a)-1]$ asymptotically for large vectors a. *The conjectured behavior of the density* $p(N)$ *at a point* $N < K(a)$ *is*

$$p(N) = \left(\frac{N}{K(a)}\right)^{n-1}.$$

Such a distribution would immediately imply that the semigroup $S(a)$ *occupies* $1/n$*-th of the segment* $[0; K(a)-1]$:

$$\int_0^K (N/K)^{n-1}\, dN = K/n$$

(*the triangle fills one half of the rectangle, the parabolic triangle fills one third, and so on*).

1999-11. Consider the density of the semigroup $S(a)$ with multiplicities taken into account (each point is counted as many times as it has representations in the form $\langle k, a\rangle$ with $k \in \mathbb{Z}_+^n$).

Find this density $P(N)$ asymptotically for large vectors a. *The conjecture is that* $P(N) \sim N^{n-1}$ *fot all* N (*rather than only for* $N < K(a)$). *Note that, for* $n = 2$, *both densities* (*taking and not taking account of multiplicities*) *asymptotically coincide for* $N < K(a)$. *It is not clear whether such a coincidence takes place for* $n \geq 3$.

1999-12. Complexify the group $\mathbb{Z}$ of integers employing the fact that $\mathbb{Z}$ is a braid group for two threads and, simultaneously, a dyed braid group for two threads. *The conjectured alternatives are* $\mathbb{Z}$ *and* $\mathbb{Z}^2$.

1999-13. Reflection groups and oscillatory integrals. Consider the oscillatory integral

$$I(h,\lambda) = \int_{\mathbb{R}^n} e^{iF(x,\lambda)/h} \varphi(x)\, dx, \quad F : (\mathbb{R}^n \times \mathbb{R}^m, 0) \to (\mathbb{R}, 0), \quad h \to 0,$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function concentrated in a sufficiently small neighborhood of the origin. The *singularity index* β of a singularity of the function $F(\cdot, 0)$ at 0 is the infimum of the numbers γ such that

$$|I(h,\lambda)| \leq C(\varphi)|h|^{\frac{1}{2}n-\gamma}$$

at all sufficiently small $|\lambda|$ under an arbitrary deformation of F (the value $\beta - \frac{n}{2}$ is then called the *oscillation index*).

For simple singularities, the singularity index equals

$$\beta = \frac{1}{2} - \frac{1}{N}, \tag{1}$$

where N is the *Coxeter number* of the corresponding Coxeter group (which is a finite irreducible group generated by reflections in $\mathbb{R}^\mu$); see ARNOLD V. I. Remarks on the stationary phase method and Coxeter numbers. *Russian Math. Surveys*, 1973, **28**(5), 19–48:

Singularity	A_μ	D_μ	E_6	E_7	E_8
β	$\frac{\mu-1}{2(\mu+1)}$	$\frac{\mu-2}{2(\mu-1)}$	$\frac{5}{12}$	$\frac{4}{9}$	$\frac{7}{15}$
N	$\mu+1$	$2(\mu-1)$	12	18	30

Formula (1) is also valid for boundary singularities (recall that the Coxeter number of B_μ equals 2μ).

Problem: Construct a theory of oscillatory integrals and find a similar formula for the remaining (noncrystallographic) Coxeter groups F_4, G_2, H_3, H_4, and $I_2(p)$ (whose Coxeter numbers are 12, 6, 10, 30, and p, respectively).

1999-14. Consider a family of smooth surfaces $z = f_t(x,y)$ in $\mathbb{R}^3$. Suppose that the surface corresponding to $t = 0$ is convex and, at some $t = t_\star > 0$, a hyperbolic region arises. In the computer experiment performed by A. Ortiz-Rodriguez, the line formed by the inflection points of the asymptotic curves in the hyperbolic region (*tacnodal line*) at small $t - t_\star > 0$ had the shape of the figure eight tangent to the boundary of the hyperbolic region at two singular points. Construct a rigorous theory of such figures eight.

1999-15. Products of matrices can be calculated by Strassen's fast matrix multiplication formula (for example, multiplying two 2×2 matrices by this formula involves 7 rather than 8 multiplications). How is this formula related to the trinity $\mathbb{R}$–$\mathbb{C}$–$\mathbb{H}$?

1999-16. On the plane $\mathbb{R}^2$, consider a configuration of n curves diffeomorphic to straight lines. It is assumed that no two curves intersect at more than 1 point, and that all intersections of the curves are transversal.

What configurations of curves are realized by straight lines? Starting from what number n of curves do deviations occur?

1999-17. Definition. The plane curve $\{x, y \mid x^{-2} + y^{-2} = 1\}$ is called an *anticircle*.

Theorem 1. The curve projectively dual to the anticircle is the astroid $\{p, q \mid p^{2/3} + q^{2/3} = 1\}$.

Theorem 2. The set of normals to an ellipse is the anticircle.

Question 1. Is there an astroid among the equidistant curves of an ellipse? *According to F. Aicardi, among the equidistant curves of an ellipse there are no curves orthogonally equivalent to the astroid. Furthermore, according to M. E. Kazarian and R. Uribe, there are no curves either projectively or affinely equivalent to the astroid*[1]. *Moreover, they proved the following*: *consider the affine transformation sending the four cusps of the equidistant curve to the four verticies of a fixed square. In the family obtained this way there is exactly one curve with the symmetry of a square. This only "candidate" tends to the astroid as the eccentricity of the ellipse tends to zero.*

Question 2. Are there multidimensional analogues of the anticircle and the astroid?

Question 3. Move the tangents to the ellipse along the normals at distance s. What are the properties of the resulting curve in the dual plane? *According to F. Aicardi, it has no cuspidal points.*

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[1] In the Russian edition of this book, the contrary was affirmed here, which was an error.