

1998

1998-1. Combine É. Cartan's theory of differential systems with singularity theory. *One should distinguish two higher (and infinite) codimension exceptions from the generic cases studies, while in the present form the Cartan theory, like the algebraic geometry, is more interested in "general statements" (like the Hilbert finiteness theorem) admitting no exceptions (in the analytic category), while to extend these to the smooth categories one should distinguish different degrees of degeneration, whose representatives might behave quite differently.*

1998-2. Consider a generic smooth surface in the three-dimensional real projective space. Can this surface have less than six special parabolic points (where the asymptotic direction is tangent to the parabolic curve, and the dual surface has a swallowtail)? If the number of special points is less than six, can the number of connected parabolic curves be less than four? *The number of special points in Panov's example with only one parabolic line is equal to 12.*

1998-3. Consider a smooth parabolic curve of constant multiplicity on a surface in the projective 3-space. Into how many parabolic curves of multiplicity one does it decompose under a small generic perturbation?

The question is open even for the surfaces whose equations in affine coordinates have the form $z = f(x, y)$ and whose parabolic curves are the infinitely distant straight line. In this case, the multiplicity is even (and equals two in the simplest situation), and the conjectured number of multiplicity-one parabolic curves of the perturbed surfaces is three.

Is it true that a general surface close to the surface specified by the equation $z = 1/(x^2 + y^2)$ [$z = x/(x^2 + y^2)$] in affine coordinates has not less than three [respectively, two] parabolic curves in a neighborhood of infinity?

1998-4. The *spherical second differential* of a function on the sphere is the quadratic form on the tangent space that measures the difference between the given function and the nearest restriction to the sphere of a function linear in the ambient space.

A function is called *hyperbolic* if its spherical second differential is hyperbolic everywhere except at finitely many points (where the function can have singularities).

Can an odd hyperbolic function on the 2-sphere have less than six logarithmic poles?

Can an odd function obtained from an odd hyperbolic function on the 2-sphere by a generic smoothing have less than eight parabolic curves (along which the spherical second differential degenerates)?

1998-5. Does there exist a surface $z = f(x, y)$ whose Gaussian curvature [at every point $(x, y, f(x, y))$] is the given function $g(x, y)$? Here, f and g are functions smooth in a neighborhood of the given point.

Answering a similar question for the Hessian requires solving the equation $f_{xx}f_{yy} - f_{xy}^2 = h(x, y)$, where h is a function defined in a neighborhood of the given point.

What singularities can a parabolic curve ($h = 0$) of a smooth surface have in a neighborhood of its flattening point (where $df = 0$ and $d^2f = 0$)?

1998-6. Let us consider a curve specified by the equations $x = \cos t$, $y = \sin t$, $z = \cos 3t$. This curve has six flattening points (of zero torsion). Is it possible to annihilate all these flattening points by an admissible regular homotopy of a curve?

A regular homotopy is called *admissible* if, in course of the deformation, there are no events of

- a) self-intersections of the curve (changes of the knot type);
- b) self-intersections of the dual curve (formed by osculating planes of the initial curve in the dual space).
- c) inflection points (zero-curvature points);
- d) tangencies of the dual curve with the surface of the front (formed by tangent planes to the initial curve).

1998-7. The curves admitting a convex projection in the projective space form a domain in the space of curves. Examine the boundary of this domain, namely, its stratification, the singularities of the intersection of the boundary with the transversals to its strata, and the complex of its strata.

Similar questions for convex curves themselves and for “strongly convex” Barner curves (see problem 1996-9) are also interesting.

1998-8. Study the cohomology rings of the complements of the bifurcation diagrams of holomorphic functions: Is it true that these complements are the Eilenberg–MacLane $K(\pi, 1)$ spaces? What are their stable Betti numbers (and cohomology rings)?

1998-9. To an entire algebraic function, in addition to its braid group, there are related a series of the groups of its second, third, etc. braids. These groups are defined as the fundamental groups of local complements of successive discriminants,

each being the set of atypical values of the projection of the preceding discriminant along the fibers of a generic one-dimensional bundle. As the initial “discriminant,” we take the graph of the function treated as a hypersurface in the product of the domain of the function by its range fibered over the domain.

These groups, discriminants, their complements, the cohomologies of these complements, and the corresponding monodromies remain absolutely unexplored even for the simplest algebraic function $z(a)$ specified by the equation $z^n + a_1 z^{n-1} + \dots + a_n = 0$.

Even a description of generators and relations in these groups is of interest. In addition to generic projections, it is interesting to consider the sequence of projections successively forgetting a_n, a_{n-1}, \dots

1998-10. How to complexify braid theory? the Maslov index? the theory of Vassiliev’s knot invariants?

1998-11. When the number m is large, what is the behavior of the greatest multiplicity (Milnor number) of a critical point of a holomorphic function in two variables depending generically on m parameters?

1998-12. Can an asymptotic line on the surface $z = f(x, y)$ all of whose points are hyperbolic be closed?

1998-13. Does the Euler equation for an ideal fluid have new conservation laws in addition to the classical ones? Are there such conservation laws along coadjoint orbits of the group of volume-preserving diffeomorphisms of the domain?

1998-14. How can we complexify the ring \mathbb{Z} ? On the set of homotopy classes of the mappings of a Lie group into itself that leave the identity fixed, two generally noncommutative operations act: “addition,” defined by $(a + b)(g) = a(g)b(g)$, and “multiplication,” defined by $(ab)(g) = a(b(g))$.

For instance, we obtain the ring \mathbb{Z} from the group $U(1) = SO(2)$ and the field \mathbb{Z}_2 from the group $O(1)$. What is obtained from $SO(3)$? from $Spin(4)$? from other groups?

1998-15. What is the quaternionic analogue of the determinant?

1998-16. How can we complexify the notions of one-dimensional holomorphic bundle and of its connection and curvature? What becomes of the theories of quantum Hall effect and of Berry phase under such a complexification?

1998-17. Contactize the symplectic Liouville theorem on completely integrable Hamiltonian systems.

1998-18. A vector field v of divergence zero on a 3-manifold is called *Hofer* if it is the field of kernels of a 2-form having contact potential [this means that the field v is specified by the condition $i_v(d\alpha) = 0$, where $\alpha \wedge d\alpha$ nowhere vanishes].

Consider the motion of a charged point on a surface under the action of a magnetic field orthogonal to the surface. Under what conditions is the corresponding vector field on the 3-manifold of unit tangent vectors to the surface Hofer? Even the case of a nonvanishing field on the sphere with standard metric is interesting.

1998-19. The Heisenberg indeterminacy relation leads to the following conjecture. Let Γ be a closed subgroup of the commutative group of the Euclidean space \mathbb{R}^n such that the quotient space by Γ is compact (e. g., a lattice).

Suppose that a ball of radius r is contained in the complement of Γ . Then in the dual Euclidean space there is a nonzero “wave vector” k of length not exceeding c/r such that the scalar product (k, x) takes only integer values when x is in Γ .

Here c is a constant depending only on n .

1998-20. Classify the simple curve singularities in a contact space.

1998-21. The following problem about Legendrian links was communicated to me by R. Penrose.

Consider the space-time \mathbb{R}^{2+1} (with pseudo-Riemannian metric of signature $++-$ positive definite on the isochrones $t = \text{const}$).

The manifold of light rays in such a space has a natural contact structure. The rays from one point of the space-time form a Legendrian submanifold in this manifold.

The problem is to study the relation between the causality (the possibility of joining two points in the space-time by a time-like curve) and the linking of the corresponding Legendrian manifolds of dimension $n - 1$ in the $(2n + 1)$ -dimensional space of rays in the $(n + 1)$ -dimensional space-time.

1998-22. Consider an n -edge polygonal knot in \mathbb{R}^3 or in \mathbb{S}^3 . How does the minimum number of simplices in a triangulation of this space whose 1-skeleton contains the most complex n -edge knot grow with increasing n ?

1998-23 (N. A. Nekrasov). Consider the quotient space of the space of (germs of) pairs of functions with zero Poisson bracket modulo the group of (germs of) symplectomorphisms. We claim that this “manifold” has a natural symplectic structure and is endowed with a natural discriminant of complex codimension one.

The problem is to study the fundamental group of the complement of this discriminant. Is this complement an Eilenberg–MacLane space? What is its cohomology ring?

1998-24 (A. N. Varchenko). The equation $u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_xu_tu_{xt} = 0$ has the property that, if u is its solution, then so is $f(u)$. What other operators have similar invariance properties (and how can they be used to construct hydrodynamical analogues, topological invariants, topological variational principles, etc.)?

1998-25. The problem on Jordan matrices by M. L. Kontsevich. Consider the space of square complex matrices of a fixed size. Can one choose one representative of each class of conjugate matrices so that all these representatives form a collection of affine subspaces of the matrix space?