

# 1997

**1997-1.** Study the combinatorics of the bifurcation diagram of the space of real trigonometric polynomials outside the set of  $M$ -polynomials (all critical points of which are real). *For the  $M$ -polynomials of degree  $n$ , there is an explicit polyhedral model. For example, at  $n = 2$ , the bifurcation diagram reduces to an astroid with diagonals, and the model is a square with diagonals.*

**1997-2.** We define a *selector* to be a piecewise linear function in  $\mathbb{R}^n$  with coordinates  $(x_1, \dots, x_n)$ , which coincides, in every region where all the coordinates are different, with one of the coordinates. Examples are given by *Matov selectors*, defined by expressions like  $\max(x_1, x_2, \min(x_3, \max(x_4, x_5, x_6)), \dots)$  (each argument enters once).

How many selectors exist in all, and how many Matov selectors? How can we recognize whether a selector is a Matov selector?

*V. I. Matov proved that, if  $f_1, \dots, f_n$  are generic smooth functions on a manifold  $M$ , and  $S$  is a Matov selector, then the function  $S(f_1, \dots, f_n) : M \rightarrow \mathbb{R}$  is topologically equivalent to a Morse function (and described the possible indices in terms of the selector). Does any other selector satisfy this property?*

*According to calculations of F. Aicardi, the numbers of Matov selectors for  $n = 1, 2, \dots$  are equal to 1, 2, 8, 52, 472, 5 504, 78 416, 1 320 064, 25 637 824, 564 275 712, ...*

**1997-3** (A. A. Agrachev – M. Ya. Zhitomirskii). Let  $\alpha$  be a 1-form nondegenerate on the boundary of a disk and vanishing at its tangent vectors, and let  $d\alpha = \alpha \wedge \beta$ . Then  $d\beta$  necessarily vanishes somewhere. *The authors claim that this is not so for surfaces with boundary different from disks.*

**1997-4.** In the theory of wave front propagation, all deformations of a Legendrian manifold under which it remains non-self-intersecting are usually considered admissible. In real-life problems on the propagation of a co-oriented front, the front can only move forward (in the direction determined by its co-orientation) and cannot move backward. The introduction of this constraint changes the problem setting both in the theory of wave fronts and in immersion theory. For instance, we can consider the oriented graph where the vertices are classes of curves and two classes  $A$  and  $B$  are joined by an arrow from  $A$  to  $B$  if  $A$  has a representative (together with its motion forward) such that, moving forward, this representative arrives at the class  $B$ .

Calculate the part of this graph that corresponds to immersions (fronts) with small numbers of self-intersections (and, for fronts, cusps). Does there exist a perestroika of  $\mathfrak{F}$  into  $\mathfrak{F}$  (with a different co-orientation) in the class of fronts with two cusps?

**1997-5.** Is the problem of the possibility of connecting two immersions of the circle into the plane by a path in the space of immersions without direct self-tangencies algorithmically solvable? *The conjecture is that it is not solvable, because in its framework, the problem of knot equivalence can (?) be modeled.*

**1997-6** (D. A. Panov). Does a generic function exist on the plane, whose Hessian is positive in a region, bordered by a smooth connected curve, and the field of asymptotic directions  $d^2f = 0$  on this parabolic curve has only one special elliptic point? Is it true that the number of hyperbolic special points on such a curve is not less than the number of elliptic ones?

*I recall the definition of elliptic (and hyperbolic) special points on a parabolic curve.*

*The special points are the points of tangency of the asymptotic direction of the graph with the parabolic curve. Over the hyperbolic region, the field of asymptotic directions defines a two-sheeted covering surface in the manifold of the non-oriented tangent elements (each point of the hyperbolic domain of the plane is lifted to the two asymptotic directions at that point).*

*For generic functions, this surface is smoothly continued by the asymptotic directions at the parabolic points. The critical line of the projection of this surface to the plane lies above the parabolic curve.*

*The asymptotic directions at the hyperbolic points are lifted to a field of directions on the surface constructed above. This field of directions on the surface is smoothly continued to the critical line, except for those "special" points of the parabolic curve, where the asymptotic direction is tangent to this curve.*

*For generic functions a special point is a singular point (a zero) of a smooth generic vector field (in a neighborhood of the point in question on the surface constructed above). A special point can be a saddle (index  $-1$ ), and in this case is called hyperbolic, either a node or a focus (index  $+1$ ), and in this case is called elliptic.*

**1997-7** (D. A. Panov). Consider a generic smooth function  $F$  on the 2-torus. Let us construct the mapping of the torus to  $\mathbb{R}P^2$  which takes each point of the torus to the point with homogeneous coordinates  $[F_{xx} : F_{xy} : F_{yy}]$ . Is it true that every point of the projective plane has no less than four preimages under this mapping  $\mathbb{T}^2 \rightarrow \mathbb{R}P^2$ ?

*For a generic function, all the three derivatives cannot vanish simultaneously. The parabolic points are mapped to points on the zero Hessian circle  $AC = B^2$  ( $A = F_{xx}$ ,  $B = F_{xy}$ ,  $C = F_{yy}$ ). Each point on this circle indeed has not less than four preimages. This follows from the Morse inequality for functions on a circle: For each translation-invariant vector field  $a\partial/\partial x + b\partial/\partial y$  on the torus, consider the derivative of  $F$  along this field. This derivative has four critical points, which give four preimages of the point on the circle of parabolic points that corresponds to the direction of the field.*

**1997-8.** Stability of pyramids. Solutions to many problems of singularity theory (such as bifurcation diagrams or caustics) have the form of a pyramid in the 3-space whose horizontal section is (more or less) akin to a hypocycloid on the plane contracting to a point as the section plane approaches the critical “zero” position.

**Example 1.** Consider a general one-parameter family of surfaces in the Euclidean space  $\mathbb{R}^3$  that passes through the “North pole”  $N$  and contains the usual sphere (corresponding to the zero parameter value). Let us mark out the first caustic of the North pole  $N$  on each surface. On the sphere, this is the South pole  $S$ . On nearby surfaces, these caustics are small curves with four (for generic families) cusps. Together, all such caustics sweep out a surface. It has the shape of the pyramid described above.

**Example 2.** Consider a generic positive function  $F$  (magnetic field) on the plane. Let charged particles move from the point  $0$  in all possible directions on the plane at a small initial velocity  $v$ . If the function  $F$  were constant, the trajectories of the particles would be Larmor circles of small radius  $v/F$ . The corresponding phase curves would form an exact Lagrangian torus in the phase space such that its projection into the plane would have two envelopes, a degenerate inner point caustic at the initial point  $0$  and an outer caustic being a circle of radius twice the Larmor radius. The entire picture depends on the parameter  $v$ .

If  $F$  is not constant, then the interior caustic is no longer a point. It turns into a small closed envelope of the perturbed trajectories of the particles moving from  $0$  at an initial velocity of given magnitude  $v$ .

This envelope has (for a generic  $F$ ) four cusps and is small together with the initial velocity  $v$ . Let us place each envelope in the separate plane  $v = \text{const}$  in the 3-space. All these envelopes sweep out a pyramid-shaped surface.

A similar pyramid was obtained by A. A. Agrachev as the caustic of a simplest system with nonholonomic constraint in control theory (the example of magnetic field fits in this scheme).

**Example 3.** Consider the four-parameter family of trigonometric polynomials

$$F_{A,a,b,c}(t) = A \cos 2t + a \cos t + b \sin t + c.$$

The caustic of this family consists of the parameters values  $(A, a, b)$  such that the corresponding function has a degenerate critical point. This surface in the 3-space has the form of a pyramid whose horizontal sections ( $A = \text{const}$ ) are hypocycloids with four cusps, being small for small  $A$ .

**Example 4.** Consider a typical two-parameter family of functions for which 0 is a point of zero minimum [e. g.,  $H_{a,b}(x,y) = x^2 + y^2 + a(x^2 - y^2) + 2bxy + Ax^3 + Bx^2y + Cxy^2 + Dy^3$ ].

Consider the three-parameter family of vanishing cycles

$$\gamma_{a,b,c} = \{x, y : H_{a,b}(x,y) = c\}.$$

The number of vertices (extrema of the curvature) of a curve  $\gamma$  with a very small  $c$  is almost always four. However, at the point  $a = b = c = 0$  of the parameter space, a narrow tongue of the locus of curves with six vertices reaches generically the plane  $c = 0$ .

This set of curves intersects the plane  $c = \text{const} > 0$  in a small plane region bounded by a curve with six cusps, similar to a hypocycloid. As  $c$  approaches zero, this ‘‘hypocycloid’’ contracts to a point. The entire boundary of the tongue of the locus of curves with six vertices in the parameter space has the shape of a pyramid near the point  $a = b = c = 0$ .

The problem is to determine the stability of the pyramid singularities mentioned above. In all cases, the question reduces to examining families of functions on the circle.

The conjectured answers are: the caustic (and the corresponding family of functions on the circle) is stable (with respect to the analytic or smooth deformations of the condition of the problem and, respectively, to the analytic or smooth normalizing diffeomorphisms) in a ‘‘conic neighborhood’’ of the corresponding pyramid in the parameter space. This ‘‘conic neighborhood’’ of the pyramid is itself bounded by a larger pyramid with the same vertex. Such a ‘‘neighborhood’’ contracts to one point at the vertex of the pyramid (and, therefore, the diffeomorphism reducing the caustic to normal form becomes only a homeomorphism at the vertex).

**1997-9.** The mathematical trinities. In addition to the pairs (an object, its complexification) in various mathematical theories, one often encounters triples of objects. The conjecture is that it is not a coincidence, and all the triples are related by commutative diagrams. The arrows joining two such triples usually form a natural triple themselves. The problem is to verify this conjecture and to study such triples systematically.

Here are several examples of such triples:

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$E_6$	$E_7$	$E_8$
$P_8$	$X_9$	$J_{10}$
$A_3$	$B_3$	$H_3$
$D_4$	$F_4$	$H_4$
tetrahedron	octahedron	icosahedron
$6 = 2 \cdot 3$	$12 = 3 \cdot 4$	$30 = 5 \cdot 6$
$60^\circ, 60^\circ, 60^\circ$	$45^\circ, 45^\circ, 90^\circ$	$30^\circ, 60^\circ, 90^\circ$
$\mathbb{S}^1 \xrightarrow{\mathbb{S}^0} \mathcal{S}^1$	$\mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathcal{S}^2$	$\mathbb{S}^7 \xrightarrow{\mathbb{S}^3} \mathcal{S}^4$
coverings	connections	?
monodromy	curvature	??
$w_i$	$c_i$	$p_i$
usual	trigonometric	modular
polynomials	polynomials	polynomials
usual	trigonometric	elliptic
numbers	numbers	numbers
cohomology	$K$ -theory	elliptic cohomology

The symbol ? denotes a conjectural “hyperconnection”; probably the latter is some quaternionic thing turning into the connection of a fibering over complex curves in the base which, however, has many complex structures over whose curves these connections have some discordance.

The symbol ?? should denote a conjectural hypercurvature 4-form (which most probably measures the extent of violation of some generalization of the Bianchi identity by a hyperconnection).