1994

1994-1. We use the term *pseudofunction* for an immersion $\mathbb{S}^1 \to S^2$ bounding half the sphere area and homotopic to an embedding of the equator in the class of immersions such that no subloop smaller than the entire curve bounds half the sphere.

Prove that a pseudofunction intersects any equator. *Proved by A. B. Givental′ even for Lagrangian $\mathbb{R}\mathrm{P}^n$ in the symplectic $\mathbb{C}\mathrm{P}^n$.*

1994-2. Prove that the number of inflections of a pseudofunction is at least four.

1994-3. Consider the cylinder $\mathbb{S}^1 \times I$. An immersion of $\mathbb{S}^1$ into this cylinder is called a 0-*pseudofunction* if it bounds half the cylinder area and is homotopic to an embedding of the boundary of an embedded disk in the class of immersed curves bounding half the cylinder area and containing no subloops bounding such an area.

Prove that a 0-pseudofunction intersects the equator. Study the existence of four inflection points for a 0-pseudofunction. *A curve on the cylinder $x^2+y^2=1$, $|z|<1$ can be projected onto the sphere $x^2+y^2+z^2=1$ either from the center or by the horizontal radii from the points on the vertical axis of the cylinder* (*i. e., by means of the Archimedean symplectomorphism*). *The former projection transforms the inflection points on the cylinder into inflection points on the sphere. The latter transforms the inflection points into points of double tangency with projections of great circles. The perturbations of the cylinder equator that, together with the equator, bound zero area have four inflection points in both senses.*

1994-4. If a curve embedded in S^2 meets the great circle $2k$ times, then it has at least $2k$ inflection points. Find the symplectic (or contact) setting of this geometric theorem and transfer it to general Chebyshev systems.

1994-5. A curve (immersed circle $\mathbb{S}^1$) in $\mathbb{R}^{2n}$ is called *convex* if no hyperplane intersects it in more than $2n$ points (counting multiplicity). Is it true that any convex curve in $\mathbb{R}^{2n}$ has a convex projection on $\mathbb{R}^{2n-2}$? or is a projection of a convex curve in $\mathbb{R}^{2n+2}$? *A similar question for projective convex curves in $\mathbb{R}\mathrm{P}^m$ with not necessarily even m is also interesting.*

1994-6. Smooth curves in $\mathbb{R}^3$ close to plane convex curves have at least four flattening points. To give a contact formulation of this assertion (in the spirit of the Morse–Chekanov Legendrian theory), it would be useful to understand how large can a deformation be while still preserving the lower bound of four flattening points. Is it sufficient to assume that the initial curve as well as the dual curve remain trivial (embedded and unknotted) in a deformation?

1994-7. Consider the Legendrian self-linking numbers L_i of a Legendrian curve in the solid torus $ST^*\mathbb{R}^2$. Are they contact-invariant (i. e., are they preserved by the contactomorphisms of the solid torus onto itself that preserve the orientation of the basis circle and the co-orientations of the contact planes)?

Solved affirmatively by E. Giroux. A positive answer would follow from the connectedness of the contactomorphism group described above, but this connectedness is not proved. It is only proved that the contactomorphisms of the above-mentioned type cannot change the type of trivialization of a torus bundle "at infinity" ($x^2+y^2 \gg 1$) *and over the basis circle.*

1994-8. What is the analog of the Bennequin inequality for Legendrian curves in ST^*M^2?

1994-9. Does the universal Milnor fibration of surfaces for A_2 in $\mathbb{C}^3$ ($x^3+\lambda_1 x+\lambda_2+y^2+z^2=0$) have a symplectic flat connection?

For curves in $\mathbb{C}^2$*, such a connection is constructed as follows*: *an elliptic curve with a marked point is identified with a neighboring elliptic curve with a marked point by a real linear realification transformation of the covering plane which maps the basis of the initial period lattice to the basis of a close lattice.*

1994-10. How does the number of isotopy classes of plane (or spherical) curves with n double points grow? What is the distribution of these curves in the index (whether the limit distribution is the Gaussian one)?

Here is the empirical distribution for the plane curves having $n=5$: *26, 133, 290, 364, 290, 133, 26.*

1994-11. Examine the singularities of the curvature form of the natural (adiabatic) connection of the bundle of the Hermitian matrix eigenvalue manifold near the discriminant of multiple eigenvalues.

1994-12. Compare the versal deformation's curves of the mappings $(\mathbb{R},0) \to (\mathbb{R}^2,0)$ with the classification of long curve immersions on the plane: What classes are realized; what are the bifurcation diagrams (remember stabilization!); how many connected components does the complement of a bifurcation diagram have; does the smooth type of a long curve determine the connected component of the complement; what are the expressions of the values that the invariants (J^+, J^-, St, and others) take on in terms of the local algebra of the singularity; what becomes of all this theory under complexifications, that is, for the mappings $(\mathbb{C},0) \to (\mathbb{C}^2,0)$?

1994-13. Consider a particle in a magnetic field on a surface M^2. Study Legendrian divergence-free vector fields on ST^*M^2 and, in particular, their closed orbits. More generally, consider divergence-free Legendrian vector fields on $\mathbb{S}^3$ for some (standard?) contact structure. Does there exist a counterexample to the Seifert conjecture (that a divergence-free field without singular points has at least two closed trajectories) in this class of vector fields?

1994-14. Consider a particle in a magnetic field on a Riemannian manifold of an arbitrary dimension. The magnetic field is given by a closed two-form on the manifold, twisting the symplectic form of the phase space. In the case of a strong magnetic field (large curvature trajectories) apply the averaging method and, at least, formulate conjectures on topological lower bounds for the number of periodic orbits. These conjectures should generalize the theorem on the existence of $2g+2$ curves of large geodesic curvature on a surface of genus g.

1994-15. Is it true that a projective curve which does not intersect any more with its osculating hyperplanes is convex (that is, the number of intersection points of this curve with any hyperplane, counted with their multiplicities, does not exceed the dimension of the ambient space)? Investigate the number of connected components and the boundary of the manifold of convex curves in $\mathbb{R}\mathrm{P}^n$ (stratification, bifurcation diagrams, stabilization, ...). *There is a single connected component if the orientation is not taken into account* (*S. S. Anisov*).

1994-16. Prove that a curve in $\mathbb{R}\mathrm{P}^n$ that is projectively dual to a convex one is convex itself. *Proved by B. A. Khesin and V. Yu. Ovsienko, a simpler proof was given by M. E. Kazarian.*

1994-17. Find all projective curves projectively equivalent to their duals. *The answer seems to be unknown even in* $\mathbb{R}\mathrm{P}^2$.

1994-18. Examine the boundary of the manifold of Möbius curves in $\mathbb{R}\mathrm{P}^2$ (the Möbius curves are those from the connected component of the space of curves having at least three inflection points, that contains all the curves close to $\mathbb{R}\mathrm{P}^1$).

1994-19. Examine the boundary of the manifold of tennis immersions $\mathbb{S}^1 \to S^2$ (a *tennis immersion* is an immersion from the connected component of the space of immersions that halve the area and have at least four inflection points, that contains all curves halving the area and close to equator in the space of curves in S^2).

1994-20. Explore the singularities of the caustic of an ellipsoid in $\mathbb{R}^4$ (or in $\mathbb{R}^n$, $n > 4$). *Conjecturally these singularities are topologically inevitable*: *caustics of other* (*convex*?) *surfaces have not less singularities, and this is true even for the Lagrangian collapse on* $\mathbb{R}^n$ (*V. M. Zakalyukin's conjecture*).

1994-21. Is it true that any knot in $ST^*\mathbb{R}^2 = \mathbb{S}^1 \times \mathbb{R}^2$ can be realized as a Legendrian knot of an immersion $\mathbb{S}^1 \to \mathbb{R}^2$? *Yes*; *solved by A. Shumakovich.*

1994-22. Prove that a convex curve in $\mathbb{R}\mathrm{P}^{2n}$ is affine (does not intersect a hyperplane). *Proved by S. S. Anisov* (*and others*).

1994-23. Consider the front of a convex curve in $\mathbb{R}\mathrm{P}^n$ (its points are the hyperplanes tangent to the curve). Are the fronts of different convex curves homeomorphic? diffeomorphic? Describe the topology (combinatorics) of a front: find the number of connected components in the complement, and so on. This is interesting even for the simplest curve $x_k = \cos kx$, $y_k = \sin kx$ ($k = 1, \dots, n$) in $\mathbb{R}^{2n}$ (and even if the answer for other curves is different). *This problem has given rise to studies of complex and real trigonometric polynomials, the Lyashko–Looijenga–Laurent mapping, and graph combinatorics, but it is still unsolved itself.*

1994-24. Are the Poincaré series of numbers of moduli in jet spaces rational functions in the majority of local problems in analysis? *For instance, is it true for almost all* f (*that is, for all* f *not belonging to some subset of infinite codimension in the space of Taylor series*) *in the following classification problems*:

— *classification of the Riemannian* (*or Einsteinian*) *metrics* f *in a neighborhood of a point in a space modulo local diffeomorphisms of this space that leave this point fixed*;

— *classification of the vector fields* f *on a manifold in a neighborhood of a singular point of a field modulo local diffeomorphisms of this manifold that leave this point fixed*;

— *classification of the smooth mappings* $f : M^m \to N^n$ *in a neighborhood of a point* $x \in M$ *modulo local diffeomorphisms of* M *and* N *that leave* x *and* $f(x)$ *fixed*;

— *classification of the Hamiltonian vector fields* f *in a neighborhood of a singular point of a field modulo local symplectomorphisms that leave this point fixed*;

— *local classification of the second order differential equations* $y'' = f(x,y,y')$;

— *classification of the germs* f *of hyper-Kähler structures on a* $4n$*-manifold modulo local diffeomorphisms*?

Recall that the Poincaré series of numbers of moduli for a given (*local*) *object is the series* $M(t) = \sum_{k=0}^{\infty} m(k)t^k$, *where* $m(k)$ *is the number of moduli of the* k*-jet of this object* (*i. e., the dimension of the moduli space*).

1994-25. Is it possible to construct a theory of sufficient jets for expansions with logarithmic terms?

1994-26. Does there exist a minimal attractor for a system of Navier–Stokes equations whose dimension unboundedly increases as the viscosity diminishes (dim $\to \infty$ as $\nu \to 0$)?

1994-27. Is it true that the minimum dimension of an attractor of a Navier–Stokes system unboundedly increases as the viscosity diminishes?

1994-28 (Ya. B. Zeldovich)**.** Does there exist a divergence-free field **v** on a three-dimensional torus $\mathbb{T}^3$ such that a magnetic field **B** satisfying the system

$$\frac{\partial \mathbf{B}}{\partial t} + \{\mathbf{v}, \mathbf{B}\} = \mu\,\Delta\mathbf{B}, \quad \operatorname{div}\mathbf{B} = 0,$$

grows exponentially as t increases for some initial field $\mathbf{B}_0$? Is there a divergence-free vector field **v** on $\mathbb{T}^3$ which is a fast kinematic dynamo?

1994-29 (Ya. B. Zeldovich – A. D. Sakharov)**.** Does there exist a volume-preserving diffeomorphism of the three-dimensional ball B^3, whose iterations make the energy of some initial divergence-free vector field grow exponentially with the number of iterations?

1994-30. Consider a smooth function u_0 defined on the disk $x^2 + y^2 \leq 1$. Find the infimum of the Dirichlet integral

$$I[u] = \iint \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right) dx\,dy$$

over the set of all smooth functions u obtained from u_0 by an area-preserving diffeomorphism of the disk.

1994-31. Consider a dust-like gravitating medium in the standard Euclidean 3-space. Describe the singularities of the caustic hypersurfaces and the particle density in the physical space after the formation of the first caustics. Is it true that the singularities of the solution to the Vlasov–Poisson equations for generic initial distributions concentrated along generic smooth Lagrangian sections of the cotangent bundle have the same topological structure as for the Vlasov equation (where the gravitational interaction is not taken into account)? Do the density singularities in neighborhoods of points on caustics and of caustic singularities have the same orders as those for non-interacting particles?

1994-32. Calculate the asymptotic behavior of the maximum oscillation indices $\beta(p)$ and $\beta_n(p)$ encountered in general p-parameter families of oscillatory integrals of functions in n variables.

1994-33. Consider a generic analytic nearly integrable Hamiltonian system: $H = H_0(p) + \varepsilon H_1(p,q,\varepsilon)$, where the perturbation H_1 is 2π-periodic in the angle variables $(q_1,\dots,q_n)$ and where the unperturbed Hamilton function H_0 depends on the action variables $(p_1,\dots,p_n)$ generically. Let n be greater than two.

Prove or disprove the following conjecture. For any two points p', p'' on the same connected component of a level hypersurface of function H_0 in the action space, there exist orbits connecting an arbitrarily small neighborhood of the torus $p = p'$ with an arbitrarily small neighborhood of the torus $p = p''$, provided that $\varepsilon \neq 0$ is sufficiently small and H_1 is generic.

1994-34. Prove or disprove the following conjecture: An equilibrium point 0 of a general analytic Hamiltonian system is Lyapunov unstable if the quadratic part of the Hamiltonian function at 0 is neither positive nor negative definite.

1994-35. Find lower bounds for the number of periodic orbits of a charge in a magnetic field, where the motion of the charge is confined to a surface and the magnetic field is orthogonal to the surface. *Conjecturally, on a surface of genus g, a charge should generically have at least* $2g+2$ *periodic orbits. From a mathematical perspective, this is a problem about closed curves with given positive geodesic curvature on the surface. When the magnetic field is sufficiently strong, the conjecture is proved, cf. problem 1994-14.*

1994-36. Consider q vectors $(\mathbf{k}_1,\dots,\mathbf{k}_q)$ applied to the origin in the Euclidean plane such that their endpoints are the vertices of a regular q-gon. Consider the sum of q equal intensity harmonic waves with these wave vectors. If $q \neq 1, 2, 3, 4, 6$

(say, if $q = 5$), then this sum is not a periodic function (though it is quasiperiodic). *Example*: $q = 5$, $H(\mathbf{r}) = \sum_{j=1}^{5} \cos(\mathbf{k}_j, \mathbf{r})$.

Is it true that all closed components of the level lines $H = h$ that bound regions containing the origin lie in a bounded neighborhood of the origin? Does a Hamiltonian system with Hamiltonian function H have an unbounded phase curve?

1994-37. Is the problem of the stability of an equilibrium point for a vector field whose components are polynomials with integer coefficients algorithmically solvable?

1994-38. This and the following four problems are concerned with the analytical (and geometric) solvability of analytical problems.

Let us introduce sets of "feasible manifolds" and "feasible mappings" with the following properties:

— the arithmetical spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are feasible for any n;

— any rational mapping is feasible;

— the image and preimage of a feasible manifold under a feasible mapping are feasible manifolds;

— the intersection, union, and mutual complements of two feasible manifolds is a feasible manifold;

— the superposition of two feasible mappings is a feasible mapping;

— if $f(x,y)$ is a feasible function, then its derivative with respect to x and its primitive function determined by its value at some feasible point are feasible.

Now, consider an analytical problem specified by some choice of functions (components of vector fields, or Hamiltonian functions, etc.), which may depend on parameters. These functions are the *data* of the problem. A *feasible set* of the problem is a minimal feasible set containing the problem data. A problem is called *analytically solvable* if its solution is a feasible function of parameters.

Prove or disprove the following conjecture: There exist a number M and two functions N and D such that the problem of the stability of an equilibrium point 0 for a vector field in $\mathbb{R}^n$ whose components are n-th degree polynomials is not analytically solvable

a) if n and d are greater than M,

b) if $d > 1$ and n is greater than $N(d)$,

c) if $n > 2$ and d is greater than $D(n)$.

1994-39. Prove or disprove the following conjecture: The problem of the integrability of a differential equation specified by a vector field in a space of dimension $n > 1$ whose components are polynomials of degree $d > 1$ is not analytically solvable.

1994-40. Prove or disprove the following conjecture: The problem of the complete integrability of a canonical Hamiltonian system specified by a polynomial Hamiltonian of degree $d > 2$ in a space of dimension $2n > 2$ is not solvable analytically.

1994-41. Definition: A problem is *geometrically unsolvable* if there are no analytically solvable problems among the problems obtained from the given one by diffeomorphic changes in the parameter space. Conjecture: The problems mentioned in *1994-38–1994-40* are geometrically unsolvable.

1994-42. Definition: A problem involving a function as a parameter is *almost solvable* if the function space contains a decreasing sequence of exceptional submanifolds of increasing codimensions such that the problem is solvable outside each of these submanifolds. Conjecture: There are no almost solvable problems among those mentioned in *1994-38–1994-40*.

1994-43. Consider a vector field in the Euclidean space $\mathbb{R}^5$. The manifold of orbits of such a field (suitably chosen) can be made diffeomorphic to an arbitrary *fake manifold* $\mathbb{R}^4$ (that is, a differentiable manifold homeomorphic but not diffeomorphic to the vector space $\mathbb{R}^4$).

Can we obtain a fake $\mathbb{R}^4$ from a vector field with polynomial components? trigonometric? analytic? elementary? Can we explicitly write at least one such vector field?

1994-44. A *pseudoperiodic mapping* is the sum of two mappings, a linear and a periodic one. A *pseudoperiodic manifold* is a point's inverse under a pseudoperiodic mapping. Consider a pseudoperiodic (but not periodic) curve in $\mathbb{R}^n$ (with respect to the fixed period lattice $\mathbb{Z}^n$). Suppose that the rank of the linear part of the corresponding mapping is maximal (i. e., equals $n-1$). In that case, evidently, the curve contains an infinite branch (finitely distant from some straight line).

Is it true that a noncompact component of such a pseudoperiodic curve is always unique? *Solved in the negative by D. A. Panov.*

1994-45. Let $A : M \to M$ be an analytic diffeomorphism of a compact analytic manifold (e. g., of the torus $\mathbb{T}^2$). Is it true that the number of periodic points of period n of such a diffeomorphism is majorized by an exponential function of n?

It is assumed here that periodic points x are nondegenerate (i. e., that 1 is not an eigenvalue of the derivative of the mapping A^n at x). Generic diffeomorphisms A have no degenerate periodic points.

1994-46. Is it true that the number of periodic orbits of periods at most T of a polynomial vector field on a compact ball in $\mathbb{R}^m$ is majorized by an exponential function of T?

1994-47. Conjecture: The number of periodic points of a mapping of class C^∞ grows almost always not faster than some exponential function of the period.

Here "almost always" means "for almost all (in the sense of the Lebesgue measure) the parameter values in each typical family of mappings depending on sufficiently many parameters."

1994-48. Consider two compact submanifolds X^k and Y^l in a compact manifold M^m. Let $A: M \to M$ be a differentiable mapping. Consider the successive images of the manifold X under the iterations A^n of the mapping A. To measure their complexity (which grows as n increases), one studies their intersections $Z(n) = (A^n X) \cap Y$ with a fixed manifold Y. These intersections $Z(n)$ are, as a rule, smooth manifolds of dimension $s = k + l - m$.

Explore the asymptotic behavior of topological complexity $|Z(n)|$ of the manifold $Z(n)$ as a function in time n.

In particular, is it true that for manifolds and mappings of class C^∞, the topological complexity of the intersection $Z(n)$ is almost always majorized by some exponential function of time n? *As the topological complexity measure one might consider the sum of the Betti numbers, the characteristic numbers, the Morse and Ljusternik–Schnirelmann numbers, the numbers of the generators and of the relations of the fundamental group, and so on.*

1994-49. Consider two germs of holomorphic curves passing through the origin of the plane $\mathbb{C}^2$:

$$(X,0) \hookrightarrow (\mathbb{C}^2,0) \hookleftarrow (Y,0),$$

and a germ of a holomorphic mapping leaving the origin invariant:

$$A: (\mathbb{C}^2,0) \to (\mathbb{C}^2,0).$$

We shall apply the iterations of A to X and study the intersections of $A^n X$ with Y. The *Milnor number* $\mu(n)$ is by definition the multiplicity of the intersection of curves $A^n X$ and Y at the origin.

Do the Milnor numbers $\mu(n)$ admit an upper bound exponential in time n?

It is assumed here that A is a mapping of finite multiplicity and that, for each n, the curves $A^n X$ and Y do not coincide.

1994-50. Consider an algebraic filtration

$$V_1 \supset V_2 \supset V_3 \supset \cdots$$

of the space $V_1 = J^\infty$ of infinite jets of pairs of holomorphic mappings

$$f : (\mathbb{C}^k, 0) \to (\mathbb{C}^m, 0), \qquad g : (\mathbb{C}^l, 0) \to (\mathbb{C}^m, 0)$$

at the origin. The varieties V_i are algebraic subvarieties in J^∞, i. e., each of these varieties is defined by polynomial equations on a finite number of Taylor coefficients. This finite number however depends on i. The *generalized Milnor number* $\mu(f, g)$ is by definition the maximum over numbers i for which the pair (f, g) belongs to V_i.

Now consider holomorphic embeddings

$$(X^k, 0) \hookrightarrow (\mathbb{C}^m, 0) \hookleftarrow (Y^l, 0)$$

and a germ of a holomorphic mapping

$$A : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0).$$

Conjecture: The generalized Milnor numbers $\mu(n)$ of the pairs $(A^n X, Y)$ admit an upper estimate exponential in n (provided that A is a mapping of finite multiplicity, and that all its Milnor numbers are finite).

1994-51. Infinitesimal version of the Hilbert 16th problem. Assume that a polynomial vector field on the plane admits a first integral whose level curves are cycles (filling at least some annulus in the plane). Consider small polynomial perturbations (of prescribed degree) of this vector field. The location of the limit cycles appearing in this perturbation is given in the first approximation by zeros of a certain integral (found by Poincaré) along nonperturbed closed curves (which are the level curves of the first integral).

Is the number of zeros of the Poincaré integral bounded (by a constant depending only on the degree of the perturbation)?

1994-52. A partial case of the previous problem: consider the full Abelian integral

$$I(h) = \oint (P\,dx + Q\,dy)$$

along an oval of an algebraic curve $H(x, y) = h$. The polynomials $P(x, y)$ and $Q(x, y)$ represent an infinitesimal variation of the Hamiltonian vector field, and $I(h)$ is the Poincaré integral. Find an upper bound for the number of real zeros of the function I for all polynomials (P, Q) of a fixed degree.

1994-53. Materialization of resonances in holomorphic dynamics. Consider a holomorphic mapping of a neighborhood G of the circle $\mathbb{S}^1$ (in the complex plane $\mathbb{C}$) onto another neighborhood of the same circle:

$$A : (G, \mathbb{S}^1) \to (G', \mathbb{S}^1).$$

Suppose that A induces a diffeomorphism of the circle $\mathbb{S}^1$ conjugate to rotation R_λ through the angle $2\pi\lambda$, the conjugating diffeomorphism B being holomorphic in some neighborhood of the circle: $A = BR_\lambda B^{-1}$. Assume that the Poincaré rotation number λ is irrational.

Suppose that the maximal disc M (diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$), where the mapping A is conjugate to the rotation, is contained in the neighborhood G of the circle $\mathbb{S}^1$ together with its boundary ∂M.

Is it true that any neighborhood of each point of the boundary ∂M contains a point in a periodic orbit of the mapping A, this orbit lying in an arbitrarily small neighborhood of the boundary? Is this true at least generically?