1990

1990-1. Let $A\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a germ of a mapping of finite multiplicity holomorphic in a neighborhood of 0. Let also X and Y be complex straight lines (or holomorphic curves) passing through 0. Let μ denote the intersection multiplicity. Is the multiplicity $\mu(A^n X, Y)$ majorized by an exponent of n? *All the multiplicities* $\mu(A^n X, Y)$ *are assumed to be finite.*

1990-2. Translate the classification of umbilical points into the language of symplectic toplogy of Lagrangian singularities (possibly optical) and at least formulate conjectures on their topological necessity.

1990-3. The caustic of a point on the convex sphere S^2 (the manifold of points conjugate to the initial point along the geodesics from this point) is naturally partitioned into connected components (of its preimage in the tangent space at the initial point under the geodesic exponent mapping). We can partition it into the first caustic (generated by the first conjugate points), the second, and so on.

Can we divide the caustic of a point on S^3 (or S^n) into infinitely many connected components? For example, under a sufficiently small perturbation of the standard metric of the sphere S^3, the first N components having the form of a double sphere S^2 apparently give precisely N connected components, each consisting of two copies of S^2 attached to each other at several (how many?) conic points (of type D_4). But it is not improbable that, starting with some (very large) N, these two-sphere components begin to merge (I know no examples!) or even form infinite chains (all the more, no examples!), even if the perturbation is very small. Maybe, it is easier to obtain examples on S^n rather than on S^3 [when each pair of spheres S^2 is replaced by $n-1$ copies of S^{n-1}; by the way, the precise arrangement of the D_4-point bridges connecting these copies (note that the D_4 points form a set of codimension 2 on S^{n-1}) is not calculated even in the framework of the first approximation of perturbation theory; this question is apparently related to caustics (focal sets) of ellipsoids in $\mathbb{R}^4$].

1990-4. A hypersurface in $\mathbb{R}\mathrm{P}^n$ is k-quasiconvex if, at each point, its second quadratic form has constant signature $\{k,l\}$, where $k+l=n-1$ (the set $\{k,l\}$ is not ordered: the hypersurfaces are not co-oriented and may be non-co-orientable!). For $k=0$, this is the usual convexity.

Is it true that any (connected) k-quasiconvex closed hypersurface embedded in $\mathbb{R}\mathrm{P}^n$ is disjoint from certain subspaces in $\mathbb{R}\mathrm{P}^k$ and $\mathbb{R}\mathrm{P}^l$ (this is so for $k=0$)?

1990-5. What topological invariants of a submanifold in a Euclidean space do admit an upper bound in terms of the complete absolute curvature (the volume of the manifold of tangent planes of this submanifold in the Grassmannian bundle over the Euclidean space)?

The sum of Betti numbers can be estimated, and so can the Morse number, while the lengths of relations in the fundamental group, apparently, cannot! *It seems that the set of admissible homotopy types of submanifolds whose complete curvatures fall in a fixed range is infinite* (*at what* (*co*)*dimensions of the submanifold and the space*?).

1990-6. Prove that a typical Hamiltonian system on the torus with pseudoperiodic Hamiltonian $ap+bq+$(periodic function) having critical points involves mixing. *Solved by K. M. Khanin and Ya. G. Sinaĭ.*

1990-7. Consider a family of analytic diffeomorphisms $x \mapsto x+a+bf(x)$ of the circle, where f is a periodic function. Is the multiplicity of periodic points arising at infinitely small b bounded (uniformly with respect to a)?

1990-8. Two conducting (k-dimensional) surfaces with potential difference 1 move toward each other (in $\mathbb{R}^n$) until the distance between them becomes ε (the charge distribution is electrostatic). Determine the asymptotic behavior of the force of attraction between the surfaces in terms of the singularities of their tangency at $\varepsilon = 0$ (*for a pair of cylinders in* $\mathbb{R}^3$*, this is a problem of A. D. Sakharov*).

1990-9. Give a precise meaning to the assertion (of M. Berry) that the asymptotics of an oscillatory integral, after all terms polynomial in the wave length are subtracted, exhibits exponentially small "jumps" of the universal form erf.

1990-10. Make a precise sense of the statement (due to V. V. Fock) that the asymptotics of slowly decaying eigenfunctions in the problem on small diffusion in a potential dynamical system with several attractors $[u_t + (uv)_x = \varepsilon\Delta u,$ $v = -\operatorname{grad} U]$ have "jumps" of the universal form erf at the borders of the attractor basins.

1990-11. Give the exact meaning of the statement (of A. D. Sakharov) affirming that the average number of vertices of the polygonal pieces, into which a planar domain is divided by many lines, is equal to 4. Generalize it to the multidimensional case. *According to F. Aicardi, the mean number of faces of any dimension of the pieces in* $\mathbb{R}^n$ *is the same as for the n-dimensional cube. But a rigorous probabilistic proof seems to be lacking.*

1990-12. Consider the manifold of non-negative functions on a manifold M. Study the singularities of this manifold (stratification, stabilization, bifurcation diagrams, homological properties of the stratification, reconstruction of M). *A generic point of the boundary is a function with a single Morse minimum. The manifold of such functions is fibered over M with a contractible fiber.*

1990-13. Study the singularities of the boundary of the manifold of contact structures on a (three-dimensional?) manifold and of the boundary of the manifold of contact forms for a given structure.

1990-14. The "Hopf invariant" $\int \alpha \wedge d\alpha$ or $\int \alpha \wedge (d\alpha)^n$ on a contact manifold does not require the condition $H^2 = 0$ or $\pi_2 = 0$. Therefore, on a contact manifold, one can try to define a Morse–Floer type complex in a non-simply-connected and/or higher-dimensional case, hoping to get an invariant of the contact structure.

1990-15. Does the signature of the Milnor fiber of a function in $\mathbb{C}^3$ has an expression in the form of an integral over the 3-knot of a singularity? Can we "drag over" p_1 to this 3-manifold (possibly, with the use of its contact structure)?

1990-16. Which of the knot invariants can be "diffused" to invariants of divergence-free vector fields (and, apparently, of Legendrian fields on a contact manifold)? Can one calculate the "linking" of diffused Legendrian submanifolds in higher dimensions?

1990-17. Let $f : M^n \to S^n$ be a smooth mapping of a closed manifold to the unit sphere in $\mathbb{R}^{n+1}$, and let τ be the volume element on M. Under what conditions does there exist an immersion $i : M^n \to \mathbb{R}^{n+1}$ such that $f = g \circ i$, where g is the Gauss mapping, and τ coincides with the volume element of the Riemannian metric on M induced (via the immersion i) by the Euclidean metric on $\mathbb{R}^{n+1}$?

1990-18. Find the group $\pi_2(G_6) = \pi_2(G_8)$, where G_n is the space of real polynomials $x^n + a_1 x^{n-2} + \cdots + a_{n-1}$ having no real roots of multiplicity higher than 2.

1990-19. Let X be one of the types of critical points of holomorphic functions which forms a set of codimension k in the space of functions in k variables. By an *inflection point of type X* of a hypersurface in a projective space we shall mean a point at which the pair (hypersurface, its tangent hyperplane) is diffeomorphic to the pair (graph of the function, its tangent hyperplane) at a critical point of type X. Let also Y be a type of critical points of functions in $k+1$ variables. Find the number of inflections of type X on a level hypersurface of a generic function of type Y (in $k+1$ variables) "vanishing" (i. e., merged) at the critical point.

1990-20. Let f be a germ of a C^∞-mapping of a real space onto itself at a fixed point of finite multiplicity. Assume that this point is a fixed point of finite multiplicity for all the iterations f^n of the mapping f. Is it true that the multiplicity of this fixed point for the iteration f^n is majorized by some exponential function ae^{cn}?

1990-21. Is it true that the number of isolated cycles of periods $< T$ of an analytic vector field on a compact manifold is majorized by some exponential function ae^{cT}?

1990-22. Describe the neighborhoods of Riemannian spheres in holomorphic surfaces with positive self-intersection numbers.

1990-23. An algebraic *correspondence* of an algebraic curve to itself is an algebraic curve in the Cartesian product of the initial curve with itself. The *discrete invariants* of such a correspondence are the genus of the initial curve, the genus of the correspondence, and the "bidegree" of the correspondence (i. e., the intersection numbers of the correspondence and the factors). Suppose that a correspondence is the graph of a diffeomorphism of the circle in a real domain. Is it true that the number of isolated cycles of this diffeomorphism is bounded above by a constant depending only on the aforesaid discrete invariants?

1990-24. How large can the number of isolated zeros of the complete Abelian integral

$$I(h) = \oint_{\gamma_h} (P\,dx + Q\,dy)$$

be, where γ_h is a closed component of the level curve $\{(x,y) : H(x,y) = h\}$, if P, Q, H are polynomials of given degrees?

1990-25. Let g be a natural number ≥ 2 and $U(x)$ a fixed polynomial of degree $2g+2$. Consider the family of hyperelliptic integrals of the first kind,

$$I(h) = \oint_{\gamma_h} \frac{P(x)}{y}\,dx,$$

where γ_h is a closed component of the level curve $\{(x,y) : y^2 + U(x) = h\}$, and $P(x)$ an arbitrary polynomial of degree $\leq g$. Is this family of integrals a Chebyshev one (i. e., is it true that for any P the number of isolated zeros of the function I is at most $g-1$)?

1990-26. A *full flag* in $\mathbb{R}^n$ consists of vector subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{R}^n$$

of all dimensions. Two flags are called *transversal* if their constituent subspaces of complementary dimensions are transversal. The set of flags not transversal to a given flag is called the *trail* of this flag. Find the number of connected components into which the trail of a flag divides a neighborhood of this flag.

1990-27. An ovaloid in $\mathbb{R}^n$ (that is, a closed hypersurface bounding a convex body) is said to be *algebraically integrable* if the volume cut off by a hyperplane from this ovaloid is an algebraic function of the hyperplane. Do there exist algebraically integrable smooth ovaloids different from ellipsoids in $\mathbb{R}^n$ with *odd n*?

1990-28. Since Poincaré, a "nonintegrable dynamical system" is usually understood to be a dynamical system having no analytic first integrals. However, we can suggest a number of other meanings of the term non-integrability, such as

1) the absence of invariant hypersurfaces (principal ideals),

2) the absence of invariant closed 1-forms (multivalued first integrals),

3) the absence of invariant distributions of tangent subspaces (invariant Pfaffian modules),

4) the absence of invariant foliations (invariant completely integrable Pfaffian systems).

Consider a dynamical system with discrete time (a diffeomorphism on a compact manifold) and an object of one of these types (function, ideal, closed 1-form, ...). The images of this object under the iterations of the diffeomorphism can form a finite set (if they are periodically repeated) or an infinite set, and they can generate a finite- or infinite-dimensional space. These properties reflect the "degree of chaos" in the dynamical system. Prove the non-integrability (in the sense of each of the four definitions given above) of all dynamical systems from some open set in the space of dynamical systems on manifolds of sufficiently high dimension.