1988

1988-1. Classify the singularities of contact-Poisson structures.

1988-2. What is the maximum difference between the number of maxima and the number of minima for an n-th degree polynomial in $\mathbb{R}^2$? The same question for the $\mathbb{R}$-Morsification of singularities.

1988-3. Investigate normal forms of a quadratic cone in the contact space $\mathbb{R}^3$ ($\mathbb{R}^5$) with respect to C^∞ and analytic germs of contactomorphisms at the vertex.

The question is related to the theory of wave transfomation and relaxation oscillations (*see the paper*: ARNOLD V. I. Surfaces defined by hyperbolic equations. *Math. Notes*, 1988, **44**(1), 489–497; *the Russian original is reprinted in*: Vladimir Igorevich Arnold. Selecta–60. M.: PHASIS, 1997, 397–412).

1988-4. What is the maximum number of periodic orbits for the diffeomorphism of $\mathbb{S}^1$ which is determined by elliptic functions similar to $x \mapsto x + a + \varepsilon \,\mathrm{sn}\, x$?

1988-5. Find the upper bound for the Hölder exponent of a continuous ("Peano") mapping of the square to the cube (is $2/3$ attained?). *Solved by E. V. Shchepin.*

1988-6. Can the number of intersection points of the image of a circle—under the n-th iteration of an analytic diffeomorphism of a surface—with another (fixed) circle grow faster than any exponent of n? *Solved by O. S. Kozlovskiĭ.*

1988-7. Can the number of periodic trajectories of a real analytic mapping of a surface to itself grow faster than any exponent of the period?

1988-8. Can various topological invariants of the intersection $(A^n X^k) \cap Y^l$ as well as the Milnor numbers and other local characteristics of the tangency of germs $(A^n X^k, 0)$ and $(Y^l, 0)$, given that $A(0) = 0$, grow faster than any exponent of n?

1988-9. Prove the stabilization for $\mu \to \infty$ of the homotopy type of the complement $\mathbb{R}^\mu \setminus A_k$ where A_k is the corresponding stratum of the discriminant A_μ (of codimension k).

1988-10. Prove the analogous stabilization for complexifications, $\mathbb{C}^\mu \setminus {}^{\mathbb{C}}A_k$.

1988-11. Carry over the four-vertex theorem from planar curves to curves on the sphere S^2.

1988-12. If the Jacobian of a germ of a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ is identically zero, then the mapping can be factored through a curve as $\mathbb{R}^2 \to K^1 \to \mathbb{R}^2$. Give a precise meaning to this assertion (algebraize it); for instance, begin with formal series and end by C^∞.

1988-13. Prove that, in $\mathbb{R}^{2n}$, there are no (nonconvex) algebraically integrable hypersurfaces (i. e., the volume of the part cut off by a hyperplane cannot be an algebraic function of the hyperplane). *Proved by Newton for* $n = 1$.

1988-14. Give a formal definition of integrability of a differential equation determined by a vector field on a manifold (the definition must be independent of the algebraic and similar structures on the manifold, i. e., the integrability property must be invariant under diffeomorphisms of the manifold). Prove the non-integrability in this sense for, e. g., typical Hamiltonian systems close to generic integrable systems.

1988-15. Transfer the four-umbilical-point theorem from surfaces to the symplectic or contact topology of Lagrangian or Legendrian singularities (prove the inevitability of D^4).

1988-16. The theory of second braids: Consider a hypersurface $\Gamma_0 : z^{\mu+1} + \lambda_1 z^{\mu-1} + \cdots + \lambda_\mu = 0$ in $\mathbb{C}^{\mu+1}$ and a sequence of projections $\mathbb{C}^{\mu+1} \to \mathbb{C}^\mu \to \mathbb{C}^{\mu-1} \to \cdots$ (along the axes $z, \lambda_\mu, \lambda_{\mu-1}, \dots$). The discriminant of the projection of the hypersurface Γ_0 onto $\mathbb{C}^\mu$ is a hypersurface Γ_1 in $\mathbb{C}^\mu$. The fundamental group of the complement of Γ_1 is the braid group.

Let us define recursively hypersurfaces Γ_k in $\mathbb{C}^{\mu+1-k}$ as the discriminants of the projections of Γ_{k-1} from $\mathbb{C}^{\mu+2-k}$ onto $\mathbb{C}^{\mu+1-k}$.

Study these hypersurfaces. Are their complements $K(\pi, 1)$ spaces? What are their fundamental groups? Can we describe the fundamental group of such a complement as the group of “Zariski relations” between Zariski relations in the preceding fundamental group?

The cases $k = 1$ (*where the braid group is described as a subgroup of the automorphism group of a free group*) *and* $k = 2$ (*where the fundamental group of the complement of the bifurcation diagram is described*) *have been examined, but the case* $k = 3$ *still remains to be studied, even for small* μ. *Though, perhaps, it would be more in spirit of the description of fundamental groups of the complement* *by Zariski relations to replace the given flag of projections by a generic flag* (*for our flag, some strata are projected on the same submanifold*).

1988-17. Consider the "stochastic web"

$$\left\{x \in \mathbb{R}^2 : \sum_{i=1}^{5} \cos(x, v_i) = c\right\}$$

(where the vectors v_i form a regular pentagon). Is it true that the diameters of this curve's closed components with interior point 0 are bounded above?

1988-18. Consider the mapping $T = AB$ of the plane to itself, where $B(x,y) = (x, y + \varepsilon \sin x)$ and A is the rotation through the angle $2\pi/5$. Consider the closed invariant curves of T bounding a domain with interior point 0. Are their diameters bounded above?

1988-19. Parametric Morse inequalities for A_3 and other singularities. Consider a generic smooth function on the space of a smooth bundle (for instance, with fiber the circle and with two-dimensional base). Over certain points of the base, the restrictions of the function to the fiber have non-Morse singularities, such as A_2 on some hypersurface in the base (on a caustic), A_3 on a stratum of codimension 2 in the base (at certain points of the base in the case of two-dimensional base, namely, at cusps of a caustic).

Study the relations between the nontriviality of a bundle (e. g., the differentials in its spectral sequence) and the inevitable singularity strata on the base (for instance, the minimum number of cusps of caustics when the base is two-dimensional).

1988-20. Consider a diffeomorphism of the boundary of a manifold to itself, which extends to a diffeomorphism of the manifold. Can this diffeomorphism be always extended as a volume-preserving diffeomorphism? What properties of the diffeomorphism of the boundary guarantee the existence of fixed points of the volume-preserving extension? *Example*: $\mathbb{S}^1 \times D^2$.

1988-21. Consider a field of directions on $\mathbb{S}^3$. Can these directions be included in planes in such a way that the obtained distribution of planes be invariant with respect to the flow of a vector field v of given direction? ($\exists \alpha, \beta : \ \alpha|v = 0$, $d\alpha = \alpha \wedge \beta$.)

1988-22. Consider a field of divergence 0 on $\mathbb{S}^3$. Does there exist a contact structure in which this field is Legendrian? Or such a structure diffeomorphic to standard?

1988-23. Transfer the construction of Pontryagin and Thom from cobordism theory to real algebraic functions. The Serre property for bundles corresponds to the possibility of covering a typical deformation of the set of real roots of a polynomial (which can vanish in pairs) by a deformation of the polynomial itself. The Pontryagin isomorphism between the homotopy groups of spheres and the cobordism groups of framed manifolds corresponds to the isomorphism between the homotopy groups of the space of functions with moderate singularities and the cobordism groups of plane curves without horizontal inflectional tangent lines in the theory of real algebraic functions in one variable (see ARNOLD V. I. Spaces of functions with moderate singularities. *Funct. Anal. Appl.*, 1989, **23**(3), 169–177; *the Russian original is reprinted in*: Vladimir Igorevich Arnold. Selecta–60. Moscow: PHASIS, 1997, 455–469). This example suggests that the similarity extends much farther and can be formalized as the corresponding calculus of singularities. This similarity had first been explicitly mentioned and used in ARNOLD V. I. Braids of algebraic functions and the cohomology of swallowtails. *Uspekhi Mat. Nauk*, 1968, **23**(4), 247–248 (in Russian); *reprinted in*: Vladimir Igorevich Arnold. Selecta–60. Moscow: PHASIS, 1997, 125–127, and especially in ARNOLD V. I. Cohomology classes of algebraic functions invariant under Tschirnhausen transformations. *Funct. Anal. Appl.*, 1970, **4**(1), 74–75; *the Russian original is reprinted in*: Vladimir Igorevich Arnold. Selecta–60. Moscow: PHASIS, 1997, 151–154.

1988-24. Quadratic forms in the Euclidean space $\mathbb{R}^n$ having a multiple eigenvalue constitute a variety of codimension two in the space of all the forms. Can one represent the corresponding discriminant as the sum of squares of two functions (polynomials, power series)? *This is so for* $n = 2$.

For the Hermitian case, the codimension (and the number of squares?) is three. For the hyper-Hermitian case of $\mathrm{SU}(2)$-invariant quadratic forms in $\mathbb{R}^{4n}$, the codimension is five.

1988-25. Consider a (possibly, anti-) commutative graded ring (or, better, an $\mathbb{R}$- or $\mathbb{C}$-algebra) with Poincaré series $1+t+t^2+\cdots$ (having one additive generator of each degree). Classify such rings (algebras) with given degrees of multiplicative generators.

In the simplest nontrivial case of a commutative ring with three multiplicative generators of degrees 1, 2, *and* 3, *the number of such algebras is* 5. *In the* *general case, it is not clear for what sets of degrees the object is simple* (*admits no moduli*): *presumably, this is always so for three multiplicative generators.*

1988-26. The eccentricity of a Hilbert space. Let $R(N)$ be the minimum number such that N balls of radius $R(N)$ can cover the unit ball in $\mathbb{R}^n$, and let $r(N)$ be the maximum number such that N balls of radius $r(N)$ contained in the unit ball in $\mathbb{R}^n$ can be disjoint. As N increases, the ratio $R(N)/r(N) = \rho(N)$ tends to a limit ρ called the *eccentricity* of the space $\mathbb{R}^n$. Examine the asymptotic behavior of the eccentricity as the dimension n increases. *Possibly,* $\lim\limits_{n\to\infty} \rho = \sqrt{2}$.

1988-27. Let $K : \mathbb{T}^2 \to \mathbb{R}_+$ be an arbitrary smooth positive-valued function on a Riemannian torus. Consider the motion of a charged particle on this torus in the presence of a magnetic field K normal to the torus, i. e., its motion along curves on the torus such that their geodesic curvatures at each point are a prescribed (for this point of the torus) positive number K.

Suppose that the metric on the torus is flat. The motion of the particle (at velocity 1) is described by a curve in $\mathbb{T}^3 = T_1\mathbb{T}^2$. The standard metric determines a parallelization, namely, the decomposition $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{T}^2$. The positivity of the curvature K implies that the phase curves on $\mathbb{T}^3$ are transversal to the fibers $\{\varphi\} \times \mathbb{T}^2$. Thus, we obtain a Poincaré mapping of the fiber $\{0\} \times \mathbb{T}^2$ to itself. This mapping is a symplectomorphism homologous to the identity symplectomorphism for a suitable symplectic structure on $\{0\} \times \mathbb{T}^2$.

Prove that such a Poincaré mapping is homologous to the identity mapping also in the case of motion on a torus with an arbitrary Riemannian metric close to flat.

1988-28. Prove that, in the situation considered in the preceding problem, the Poincaré mapping is homologous to the identity mapping for a motion on the torus $\mathbb{T}^2$ with an arbitrary Riemannian metric provided that the geodesic curvature K is sufficiently large.

1988-29. Consider the torus $\mathbb{T}^2$ with an arbitrary metric and an arbitrary positive-valued function K on $\mathbb{T}^2$. Does there exist a Poincaré mapping or even a surface transversal to the vector field of the motion of a charged particle on $\mathbb{T}^2$ in the magnetic field K and isotopic to a section of the bundle $T_1\mathbb{T}^2 \to \mathbb{T}^2$?

1988-30. Prove the existence of the expected number of closed trajectories of the motion of a charged particle in a magnetic field on an arbitrary surface, at least in the cases where the field K is sufficiently strong or where the metric is close to that of constant curvature. *I believe that it is expedient to directly apply the "hyperbolic Morse theory" rather than to reduce the problem to examining fixed points of a symplectomorphism. In the case of a sufficiently strong magnetic field K, this conjecture is proved*: *the number of closed orbits is not less than* $2g+2$ *on surfaces of genus* g; *cf. problem 1994-14.*

1988-31. Generalization of the preceding problem: Consider a nontrivial bundle $M^3 \to N^2$ with fiber $\mathbb{S}^1$ endowed with a connection (specified by a field of two-dimensional planes transversal to the fibers). Let τ denote some volume element on M^3, and let v be a divergence-free (with respect to τ) vector field transversal to the plane of the connection. Is the number of closed orbits of such a vector field bounded below by the minimum number of critical points of functions on the surface N^2 (supposed to be an oriented surface without boundary)?

1988-32. A special case of the preceding problem: Is it true that an arbitrary divergence-free vector field on $\mathbb{S}^3$ making an acute angle with the Hopf field at each point has at least two geometrically different closed orbits?