

1976

1976-1. Given a system of Newton polyhedra, is there a system of *real* polynomials with these polyhedra which has the correct number of real roots (i. e., the same as for a system with generic complex coefficients)?

1976-2. Consider two plane polynomial vector fields of degrees m and n , respectively. Is it possible to estimate the number of intersection points of their limit cycles in terms of n and m (find a sharp attainable estimate)?

1976-3. Investigate the convergence of the normal forms of equations of the form $y'' = f(x, y, y')$.

1976-4. Build a theory of the “non-Desargues curvature form” (that measures local non-equivalence to a linear equation) for $y'' = f(x, y, y')$.

1976-5. Construct a symplectic (or contact) version of the asymptotic Hopf invariant: $H : M^{2n} \rightarrow \mathbb{R}$, ω is a symplectic structure, $\omega|_{H=h} = d\alpha$, $\beta = \alpha \wedge \omega^{n-1} \in \Omega^{2n-1}$, $\int_{H=h} \beta = \text{const}$ is a symplectic analog of the Hopf invariant. Given a Hamiltonian vector field, study how to measure the average rate of evolution of a Lagrangian subspace of the tangent space under the flow.

1976-6. Elaborate a theory of $\mathbb{C}P^1$ -neighborhoods in complex manifolds (similar to the theory of neighborhoods of elliptic curves already constructed, and preceding the theory of neighborhoods of higher genera curves).

1976-7 (A. Tresse). Justify the finiteness theorems of differential invariant theory.

1976-8. Consider a function with Newton diagram Γ . Is it true that each of its singularities of finite multiplicity is stably equivalent to a Γ -nondegenerate one?

1976-9. Classify the typical singularities of synthesis in a generic problem of optimal control given by a typical indicatrix field—a generic family of mappings of a fixed manifold into all tangent spaces to the base manifold (with the point of the base as a parameter).

1976-10. Investigate the asymptotic behavior of the measure of deviated trajectories in the problem of a generic perturbation of a generic k -frequency conditionally-periodic system with m slow variables.

1976-11. For a given plane vector field with a singular point, construct an algebraic complex whose homology describes the limit cycles vanishing at the singular point.

1976-12 (A. G. Kushnirenko). The Descartes rule implies that the number of real roots of a polynomial has the number of its monomials as an upper bound. Extend this observation to polynomials of several variables: the simplicity of a formula implies bounds on the topology of the variety defined by it. *A theory of “fewnomials” has been elaborated by K. A. Sevast'yanov and A. G. Khovanskiĭ, but the estimates obtained in the multidimensional case probably are strongly nonsharp.*

1976-13. Is the stratum $\mu = \text{const}$ smooth? *The smoothness of the stratum would follow from an affirmative solution to the following question.*

Given an algebraic action of a complex algebraic group on a finite-dimensional affine space (e. g., a linear representation), consider the set of all points whose stationary subgroups have a fixed dimension. Is this set a smooth manifold?

1976-14. Does the real modality of a real-valued function coincide with the complex one?

1976-15. For which weights $\alpha_s = A_s/N$ does there exist a nondegenerate quasi-homogeneous function of degree 1?

1976-16. Evaluate the modality of a Γ -nondegenerate function in terms of its Newton diagram Γ . In particular, prove that for semi-quasihomogeneous functions the modality is equal to the number of monomials in a basis of the local ring on the diagram and above it.

1976-17. Evaluate the signature of the quadratic form defined by the intersection index in the middle-dimensional homology of a local nonsingular level set of a Γ -nondegenerate function of n variables when $n \equiv 3 \pmod{4}$.

1976-18. Find a normal form for all the Γ -nondegenerate functions with a given Newton diagram Γ .

1976-19. Find the Jordan canonical form of the monodromy operator of a Γ -nondegenerate function with a given Newton diagram Γ .

1976-20. Let f be the Morsification of a real-valued function with a given singularity (say, Γ -nondegenerate). What is the maximum number of components a local real level set of f can have?

1976-21. Chebyshev polynomials of several variables. With every critical point of a function of finite multiplicity, one can associate a ‘‘Chebyshev polynomial,’’ which is the Morsification of this function with the least possible number of critical values. (The usual Chebyshev polynomials come from one-variable functions of the form z^n .) Which of the nice properties of Chebyshev polynomials in one variable hold for the above defined polynomials of several variables?

1976-22. Uniform estimates for oscillatory integrals. An *oscillatory integral* has the form

$$I(h, \lambda) = \int_{\mathbb{R}^n} e^{iF(x, \lambda)/h} \varphi(x) dx, \quad h \rightarrow 0, \quad (1)$$

where φ is a smooth function concentrated at a sufficiently small neighborhood of the origin; F is a real-valued deformation of the function $f = F(\cdot, 0)$ depending smoothly on the parameter λ ; h is a small parameter; $F(0, 0) = 0$. The *uniform index* β of the singularity of the function f in the point 0 is the infimum of γ such that for any deformation F

$$|I(h, \lambda)| \leq C(\varphi) |h|^{\frac{1}{2}n - \gamma} \quad (2)$$

for all sufficiently small $|\lambda|$.

The content of the problem is to evaluate the index β (say, for Γ -nondegenerate functions f).

For every pair of integers n and l the *universal uniform index* $\beta(n, l)$ can be defined as the infimum of γ such that the oscillatory integral (1) has an estimate (2) which is uniform in λ for all families F of functions of n variables x and l parameters λ except a meager subset in the function space.

The problem of evaluating rational numbers $\beta(n, l)$ appears to be very difficult, because it seems to be almost equivalent to the problem of the full classification of all singularities.

For a fixed l and $n \rightarrow \infty$, the numbers $\beta(n, l)$ stabilize:

$$\beta(n, l) = \beta(\infty, l) = \beta(l), \quad \text{if } n \text{ is large enough.}$$

The rational number $\beta(l)$ is the greatest singularity index among singularities of codimension l .

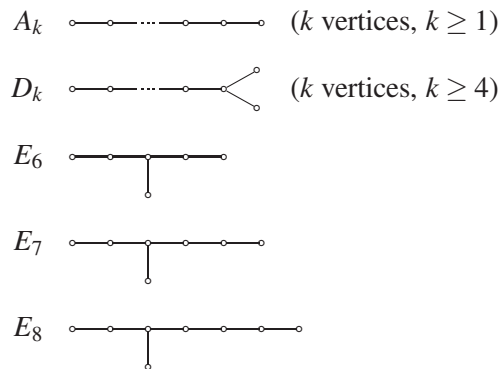
A problem for optimists: find all $\beta(l)$. A problem for pessimists: find $\beta(1000)$.

1976-23. Uniform estimates of preimage variations. Let g be the germ of a diffeomorphism $(\mathbb{R}^n, O_1) \rightarrow (\mathbb{R}^n, O_2)$ or $(\mathbb{C}^n, O_1) \rightarrow (\mathbb{C}^n, O_2)$, and let μ be the multiplicity of the point $g(O_1) = O_2$. The preimages of O_2 can merge in different ways depending on the type of the singularity of g at O_1 . Investigate the asymptotic behavior of various geometric characteristics of the preimage of a small ball of radius δ centered at O_2 as $\delta \downarrow 0$. Thus, describe the different ways in which the preimages of O_2 can merge.

An example of such a characteristic could be the so-called *variations* of all dimensions.

The variation $\sigma_k(D)$ of a sufficiently good set $D \subset \mathbb{R}^n$ is the mean value of the k -dimensional volume of the orthogonal projection of D onto a k -dimensional subspace L , over all subspaces $L \subset \mathbb{R}^n$. The volume is counted with multiplicities, i. e., the number of connected components that are projected into one point. In particular, $\sigma_n(D)$ is the volume of D and $\sigma_0(D)$ is the number of its connected components.

1976-24. The A, D, E problem. Surprisingly, the Dynkin diagrams



appear while solving various classification problems, such as the classification of:

- 1) critical points of a function;
- 2) regular polytopes (or finite orthogonal groups) in \mathbb{R}^3 ;
- 3) categories of vector spaces and linear maps;
- 4) caustics;
- 5) wave fronts;
- 6) groups generated by reflections (or Weyl groups with roots of equal norm);
- 7) simple Lie groups;
- 8) singularities of algebraic hypersurfaces with positive or negative definite intersection form of a neighboring smooth fiber.

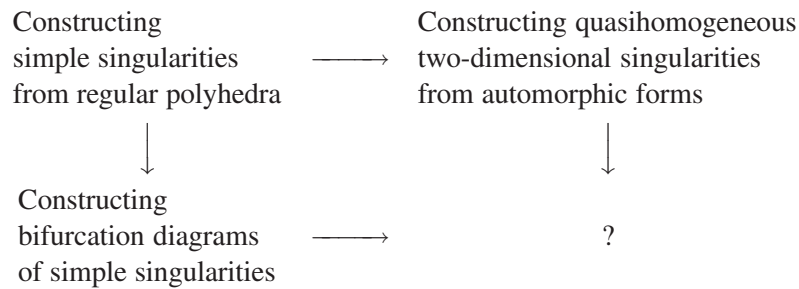
Some connections between these objects are known. However, in most cases, no explanation of the same answer for different problems has been given.

The A, D, E problem: Find a *general* classification theorem from which the solutions to all of the above problems would follow.

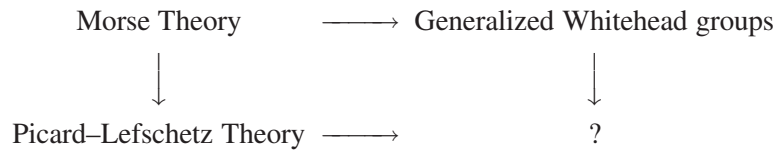
1976-25. The $K(\pi, 1)$ problem. In the case of simple singularities A, D, E the complement of the bifurcation manifolds of a function and the complement of the bifurcation varieties of its level set are the Eilenberg–MacLane $K(\pi, 1)$ spaces. Can this result be generalized to the case of nonsimple singularities?

More general problem: investigate topological properties of the complements of the bifurcation subsets of differentiable maps.

1976-26. Complete to a commutative diagram:



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1976-28. The stable cohomology ring. One can associate with a critical point of a holomorphic function f the cohomology ring $H^*(f)$ of the complement of the bifurcation diagram of the level sets in the base of a versal deformation.

Let f_2 be the germ of a function from a versal deformation of f_1 . Then the transversal to the stratum corresponding to f_2 in the base of the versal deformation defines an inclusion of the complements, and hence a homomorphism of the cohomology rings $H^*(f_1) \rightarrow H^*(f_2)$. For example, if $f_1 = x^n$ and $f_2 = x^{n-1}$ then H^* are the cohomology rings of the braid groups with n and $n - 1$ threads, respectively. Moreover, the homomorphism induces the stabilization of the cohomology rings of the braid groups as $n \rightarrow \infty$.

Does a similar situation appear in general? If yes, what would be the stable cohomology ring?

Similar questions arise for the complements of the bifurcation diagrams of functions.

1976-29. The converse of the Lagrange–Dirichlet theorem. Prove that the equilibrium 0 of a Newton system $\ddot{x} = -\text{grad}U$ is unstable whenever the critical point 0 of the polynomial (or arbitrary analytic function) $U(x_1, \dots, x_n)$ is not a point of local minimum.

1976-30 (R. Thom). Let $U : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial. Prove that at least one phase curve of the system $\dot{x} = \text{grad}U$ meets the critical point 0 with tangency to some straight line.

1976-31. Algorithmic insolvability of the problem of stability. Is the problem of stability of the equilibrium 0 of a system $\dot{x}_k = P_k(x)$ algorithmically unsolvable? Here $P_k(x)$, $k = 1, \dots, n$, are polynomials with rational coefficients.

There are closely related problems whose algorithmic insolvability might imply the algorithmic insolvability of the previous one:

1) The problem of existence of a limit cycle for the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where P, Q are polynomials with rational coefficients.

2) The problem of positiveness of a real Abelian integral $\oint R(x, y) dx$ along an oval $P(x, y) = 0$, where P, R are polynomials with rational coefficients.

1976-32. Typical singularities of solutions of variational problems. It is known that variational problems lead to discontinuities and singularities even when everything is smooth in the setting of the problem. The singularities that appear can be pathologically complex because of infinite degeneracies. Is it possible to avoid the pathology by considering generic problems?

Examples: the problem of bypassing an obstacle; the problem of the quickest path with bounded velocity $\dot{x} \in F_x \subset T_x \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$; the problem of attainable points. If we replace the indicatrix F_x with its convex hull, then we get the following problem: describe the singularities of the convex hull of a generic k -dimensional submanifold in \mathbb{R}^n .

1976-33. Singularities in the theory of partial differential equations. Consider a generic partial differential equation with smooth initial data and smooth boundary conditions. Describe the nature of singularities of solutions of the system on surfaces, curves, and at points in space, that are responsible for the situation when a solution is not smooth but belongs to some functional space.

1976-34. Compositions of algebraic functions. Consider an algebraically closed field k and n independent variables x_1, \dots, x_n over k . Let K be the algebraic closure of the field $k(x_1, \dots, x_n)$. For every natural number $r \leq n$ define the subfield $M_r \subset K$ of all elements of K that can be obtained by a composition with not more than $s - 1$ iterations of algebraic functions of r variables. Clearly, $M_1 \subset M_2 \subset \dots \subset M_n = K$.

Can it happen that $M_r = K$ for some $r < n$? More generally, how many distinct fields are there among M_1, \dots, M_n ? How many distinct fields are there among M_r^s , where $M_r^s \subset M_r \subset K$ is the subfield of all elements of K that can be obtained by a composition of not more than $s - 1$ algebraic functions of r variables. For example,

$$M_n^1 = k(x_1, \dots, x_n), \quad \bigcup_{s=1}^{\infty} M_r^s = M_r.$$

Find the minimal M_r (or M_r^s) which contains an element f satisfying

$$f^n + x_1 f^{n-1} + \dots + x_{n-1} f + x_n = 0.$$

Similar questions make sense for the field $k(x_1, x_2, \dots)$ with an infinite number of variables x_i .

1976-35. How many connected components can the complement of a degree n algebraic hypersurface in $\mathbb{R}P^k$ have? *This is unknown already for $k = 3$.*

1976-36. What are the possible arrangements of ovals of a plane projective curve of degree d such that the number of ovals is maximal possible, i. e., equal to $1 + \frac{1}{2}(d-1)(d-2)$?

1976-37. Can a planar vector field defined by two quadratic polynomials have more than 3 limit cycles?

1976-38. Determine the singularities and other analytic properties of thermodynamic functions when the interaction potential is known.

1976-39. Does a symplectic diffeomorphism of the two-dimensional torus have a fixed point whenever this diffeomorphism is homologous to the identity?

1976-40. What could be the mathematical equivalent of the physical notion of turbulence? One of the aspects of this question: find “good” theorems of existence and uniqueness for the 3-dimensional Navier–Stokes equations.

1976-41. Find mechanical (physical, chemical, etc.) phenomena which can be described by systems with exponential repulsion of trajectories and with internally unstable attracting modes.

1976-42. The numerical qualitative or ergodic investigation of multidimensional dynamical systems (and, in particular, of limit modes in these systems) relies on posing questions that are realistic, rather than those that usually appear in abstract classification theorems. The high-priority problems here are:

1) teach a computer how to determine whether a trajectory enters a neighborhood of an attracting invariant set;

2) if it does enter, teach a computer how to determine the dimension of this set and, if possible, its topology;

3) teach a computer how to find the ergodic characteristics of motion on this set; first of all how to determine whether the trajectories have exponential instability on this set (i. e., whether the entropy is positive).