1963

1963-1. Is there true instability in multidimensional problems of perturbation theory where the invariant tori do not divide the phase space?

1963-2. Prove the presence of nondegenerate hyperbolic points (and separatrix splitting) in any neighborhood of an elliptic fixed point 0 of a generic analytic area-preserving mapping $(\mathbb{R}^2, 0) \hookleftarrow$.

1963-3. Are there bounded motions filling up a set of positive measure in the three (and n) body problem, for any values of the masses and for the distances comparable with each other? Does there exist a critical value of the perturbation parameter μ at which the invariant torus with given Diophantine frequency vector breaks up?

1963-4. Let T be an orientation-preserving analytic diffeomorphism of a circle onto itself with Diophantine rotation number ω. Can one always turn T into the rotation T_0 through the angle $2\pi\omega$ via an analytic change of variables S: $STS^{-1} = T_0$?

1963-5. Consider a system of linear differential equations with quasi-periodic coefficients

$$\dot{q} = \omega, \quad \dot{x} = A(q)x; \qquad q \in \mathbb{T}^k = \mathbb{R}^k/2\pi\mathbb{Z}^k, \quad x \in \mathbb{R}^n,$$

where $\omega \in \mathbb{R}^k$ is a constant vector with Diophantine components while $A : \mathbb{T}^k \to \mathrm{gl}(n, \mathbb{R})$ is an analytic function. Is such a system always reducible for $k > 1$, $n > 1$?

1963-6. Let Γ be a (generally noncommutative) group with finitely many generators $a_1, \ldots, a_s$. By a dynamical system with the "time" Γ we shall mean an *action* of the group Γ on a space with measure Ω by measure-preserving transformations A_γ ($\gamma \in \Gamma$). For such a system, time averages may be defined as follows. Let us consider the set Γ_n of elements of Γ that can be obtained by n (but not less than n) multiplications from $a_1, a_1^{-1}, \ldots, a_s, a_s^{-1}$, and let $N(n)$ be the number of such elements. Then define the "time average" f_n of a function f as

$$f_n(x) = \frac{1}{N(n)} \sum_{\gamma \in \Gamma_n} f(A_\gamma x), \quad x \in \Omega.$$

Now let Ω be a homogeneous space, with a transitive action of a compact Lie group G on it; and let the transformations A_γ ($\gamma \in \Gamma$) belong to G.

Are the ergodic theorems of Birkhoff and von Neumann true for such dynamical systems with a noncommutative time?

The next three problems also concern dynamical systems (Ω, G, Γ) with a noncommutative time Γ.

1963-7. For some groups Γ the sequence of points $A_\gamma x$ is uniformly distributed in its closure, if the closure is connected. In other words, the time averages $f_n(x)$ of a continuous function converge to the space average over the closure $\overline{\Gamma(x)}$ of a trajectory $A_\gamma x$ $(\gamma \in \Gamma)$:

$$\lim_{n\to\infty} f_n(x) = \frac{1}{\operatorname{mes}\overline{\Gamma(x)}} \int_{\overline{\Gamma(x)}} f(y)\, d\mu(y).$$

Examples are given by the free group Γ with two generators a, b and the group Γ with generators a, b, c and the relation $abc = e$.

Does this result extend to arbitrary groups Γ with finitely many generators?

1963-8. Does the result mentioned in the previous problem extend to the noncompact case? (For instance, let Ω be the Euclidean plane or the Lobachevskian plane.)

1963-9. What is the generalization of the result mentioned in problem *1963-7* to the case where a Lie group, e. g., the isometry group of the Lobachevskian plane, is considered as time?

1963-10. In what cases is the monodromy group of the system $dx = [A(z)\, dz]x$ of linear differential equations on a Riemann surface M bounded? Here $z \in M$, $x \in \mathbb{C}^n$, and $A(z)\, dz$ is a matrix of differentials which are analytic in z except for a finite set of singular points.

1963-11. Consider a system of linear differential equations $dx/dz = A(z)x$, where $z \in \mathbb{C}\mathrm{P}^1$, $x \in \mathbb{C}^n$, and A is a matrix which depends on z analytically, except for three singular points z_1, z_2, z_3 on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Denote $\mathbb{C}\mathrm{P}^1 \setminus \{z_1, z_2, z_3\}$ by Z. If the monodromy group of the system $dx/dz = A(z)x$ is bounded, then this system has a single-valued first integral $(B(z)x, \bar{x}) = \text{const}$, where $B(z)$ is a positive definite self-adjoint matrix, single-valued for $z \in Z$.

Is it true that the surface depicting the solutions of this system in the $(2n+1)$-dimensional manifold M_c: $(Bx, \bar{x}) = c$, is uniformly distributed with respect to the following metric: on Z, we introduce a metric of constant negative curvature, and on $\mathbb{C}^n(z)$ the metric is defined by the scalar product $(B(z)x, y)$?

1963-12. The system $dx/dz = A(z)x$ from the previous problem can be considered as a dynamical system where the role of the time is played by the universal covering of Z, i. e., by the Lobachevskian plane. But an ordinary dynamical system with continuous time can also be related to this system. In order to do so, consider a new phase space whose points are the points $(z,x) \in M_c$ together with the direction ξ of a vector tangent to Z at z. The motion is defined in the following way: the point z is moving uniformly along the geodesic in the direction of ξ, and x over z is moving according to the equations $dx/dz = A(z)x$. The metric and the invariant measure are defined as in the previous problem.

This construction allows us to "multiply" the flow defined on a manifold by a group of automorphisms (which is a representation of the fundamental group of the manifold). The problem is in the study of the resulting "products."